FOREIGN COINS—THEIR BRITISH VALUE.

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Crusade—Portugal, 2s. 4d.
Dollar—Spanish, 4s. 3d.; American, 4s. 2d.
Ducat—Flanders, Sweden, Austria, and Saxony, 9s. 3d.; Denmark, 8s. 3d.
Florin—Prussia and Poland, 1s. 2d.; Flanders, 1s. 6d. Germany (Austria), 2s.
Franc, or Livre—French, 9½d.
Guilder—Dutch, 1s. 8d.; German, 1s. 7d. to 2s.
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Moidore—Portugal, 25s. 6d.
Pagoda—Asia, 8s. 9d.
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Pistole—Spain, or Barbary, 16s. 3d.; Italy, 15s. 6d.; Sicily, 15s. 4d.
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Rial—8 to a dollar, 6½d.
Rix-dollar—German, 3s. 6d.; Dutch, Hamburg, Denmark, and Sweden, 4s. 3d.
Rouble—Russian, 3s. 3d.
Rupee—Asia, Silver, 1s. 10d.
——— ———, Gold, 28s. 9d.
Sol, or Sou—French, ½d.

ADVERTISEMENT.

In the present Treatise it has been the Author's endeavour to combine what is necessary of the *Philosophy* of the *Science of Arithmetic* with the *Practice* of the *Art of Numbers:* but it is not the purpose of the work to enter into the *History* of *Arithmetic*, which has been so amply treated of in many other publications, nor to attempt any eulogium upon its merits and practical utility, which are every day so fully evinced: it is considered sufficient to place before the student an outline of the plan which has been adopted in the arrangement, with a short account of the more important divisions, leaving him to consult the *Table of Contents* for particular information respecting what may be found in their more minute details.

The *first* Chapter commences with the elementary Definitions; it then proceeds to the explanation of *Notation* and *Numeration*, which are both exemplified in a great variety of instances; and concludes with the consideration of the *Fundamental Operations* of the Science as applied to *pure* or *abstract* numerical magnitudes.

In the *second* Chapter, the Application of the Fundamental Operations has been extended to *mixed*

THE
PRINCIPLES AND PRACTICE
OF
ARITHMETIC,

COMPRISING THE

NATURE AND USE OF LOGARITHMS,

WITH THE

COMPUTATIONS EMPLOYED BY ARTIFICERS, GAGERS, AND LAND-SURVEYORS.

DESIGNED FOR THE USE OF STUDENTS.

BY

JOHN HIND, M.A., F.C.P.S., F.R.A.S.,
LATE FELLOW AND TUTOR OF SIDNEY SUSSEX COLLEGE, CAMBRIDGE.

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M.DCCC.XL.

to any one who is not engaged in *Scientific* Speculations, or in *Professional* Calculations.

It may perhaps be objected that the *Examples for Practice* given in the work, are too numerous for a rapid advancement in the subject; but the student will recollect that he has no occasion to trouble himself with the *rest*, when a *few* of them have rendered him perfect in the Application of the Rules; although it must be observed, that a *Facility* in Arithmetical Calculations is of all things the most indispensable, in the formation both of the future *Analyst*, and of the *Man of Business*.

CAMBRIDGE,
December 7, 1839.

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TABLES

OF

MONEY, WEIGHTS AND MEASURES,

WITH SOME OBSERVATIONS RESPECTING THEM.

I. Table of Money or Value.

A Farthing is written or marked $\frac{1}{4}d.$

2 Farthings are	1	Halfpenny	$\frac{1}{2}d.$
4 Farthings	1	Penny	1*d.*
12 Pence	1	Shilling	1*s.*
20 Shillings	1	Pound	£1.

Money as expressed by means of these denominations is commonly called *Sterling* money, in order to distinguish it from *Stock*, &c., which is merely *Nominal.*

The *Standard* gold coin of this Kingdom is made of a metal consisting of 22 parts of *pure gold,* and 2 parts of *copper.* The Pound sterling is represented by a gold coin called a *Sovereign,* and from a pound troy of standard gold are coined $46\frac{29}{40}$ sovereigns, so that the weight of each is 5dwts. $3\frac{171}{623}$grs., or 123.274grs.; and the *Mint* price of standard gold is therefore very nearly £3. 17*s.* $10\frac{1}{2}d.$ per ounce.

The *Standard* silver coin consists of 37 parts of *pure silver* and 3 parts of *copper,* and a pound troy of this metal furnishes 66 *shillings,* so that the weight of a shilling is 3dwts. $15\frac{3}{11}$grs., and the *Mint* price of standard silver is 5*s.* 6*d.* per ounce. The silver coinage is not a *legal tender* for more than 40*s.*, the gold coinage above mentioned being the only *general* standard of value.

In the copper coinage, 24 pence are made from an avoirdupois pound of copper, so that a penny should weigh $10\frac{2}{3}$drs. avoirdupois, or $291\frac{2}{3}$grs. troy: but this is not a legal tender for more than 12*d.*

A *Farthing* is the lowest denomination in use, but it is customary to denote farthings by *Fractions* of a Penny as in the table.

Though all Commercial Transactions are conducted by means of the Money enumerated in the preceding table, there are other coins or denominations frequently met with, and some of them more particularly in old documents, of which the following are the most important, and their values in *current* money are here annexed.

...rlk

2 = 1/2

4 = 2 = 1

48 = 24 = 12 = 1

...0 = 480 = 240 = 20 = 1 £

	£.	s.	d.
A Groat is	0	0	4
A Tester	0	0	6
A Half Crown	0	2	6
A Crown	0	5	0
A Seven Shilling Piece	0	7	0
A Half Sovereign	0	10	0
A Half Guinea	0	10	6
A Guinea	1	1	0
A Noble	0	6	8
An Angel	0	10	0
A Mark	0	13	4
A Carolus	1	3	0
A Jacobus	1	5	0
A Moidore	1	7	0
A Six-and-thirty	1	16	0

II. TABLE OF AVOIRDUPOIS WEIGHT.

A Dram is written 1 dr.

16 Drams are	1 Ounce	1 oz.
16 Ounces	1 Pound	1 lb.
14 Pounds	1 Stone	1 st.
2 Stone or 28 lbs.	1 Quarter	1 qr.
4 Quarters or 112 lbs.	1 Hundredweight	1 cwt.
20 Hundredweight	1 Ton	1 ton.

A Firkin of Butter is 4 stone or 56 lbs.; a Fodder of Lead is $19\frac{1}{2}$ cwt.: and several sorts of Silk are sometimes weighed by what is called a *great pound* of 24 ounces.

By means of this table are computed the weights of all substances of a coarse or drossy nature, as Groceries and most of the Necessaries of Life, with all Metals, except *Gold* and *Silver*. See Article (210).

Wool Weight.

7	Pounds are . .	1 Clove . . .	1 cl.
2	Cloves	1 Stone . . .	1 st.
2	Stone	1 Tod	1 tod.
$6\frac{1}{2}$	Tods	1 Wey. . . .	1 wey.
2	Weys	1 Sack . . .	1 sa.
12	Sacks	1 Last	1 la.
240	Pounds . . .	1 Pack . . .	1 pack.

III. Table of Troy Weight.

A Grain is written 1 gr.

24	Grains are . . .	1 Pennyweight . . .	1 dwt.
20	Pennyweights .	1 Ounce	1 oz.
12	Ounces	1 Pound	1 lb.

dwt
24 = 1
oz
480 = 20 = 1
lb
5760 = 240 = 12 = 1

This weight is applied to gold, silver, jewels, liquors, &c., and is generally used in Philosophical Experiments. See Article (211).

Apothecaries Weight.

20	Grains are . .	1 Scruple . . .	1 scr. or ℈.
3	Scruples . . .	1 Dram	1 dr. or ʒ.
8	Drams	1 Ounce . . .	1 oz. or ℥.
12	Ounces. . . .	1 Pound	1 lb. or ℔.

This weight is employed by Apothecaries in making up Medical Prescriptions in which the latter Symbols are generally used; and the pound is the same as the imperial pound troy. See Article (211).

IV. Table of Lineal Measure.

An Inch is written 1 in.

12	Inches are	1 Foot . . .	1 ft.
3	Feet	1 Yard . . .	1 yd.
$5\frac{1}{2}$	Yards	1 Pole . . .	1 po.
4	Poles or 22 yds.	1 Chain . .	1 ch.
40	Poles or 220 yds.	1 Furlong .	1 fur.
8	Furlongs or 1760 yds. .	1 Mile . . .	1 mi.

By this measure are computed the lineal dimensions of all magnitudes, with the exception mentioned below. See Article (208), and also Article (189), &c.

9 = 4 = 1. F
36 = 16 = 4 = 1. Ell
45 = 20 = 5 Eng 1.
27 = 12 = 3 = Fl 1.

Inches nail
$2\frac{1}{4} = 1$.

CLOTH MEASURE.

4 Nails are	1 Quarter	1 qr.
4 Quarters	1 Yard	1 yd.

qr. English Ell 5 = 1
Flemish Ell 3 = 1

This measure is used for all kinds of Cloth: and a Nail, being a *sixteenth* part of 1 yard or of 36 inches, is therefore equal to $2\frac{1}{4}$ inches. An *Ell* is 5 quarters in *England*, but the *Flemish* and *French* Ells are nearly equal to 3 and 6 *English* quarters respectively.

To these the following table may be annexed, as exhibiting the magnitudes of certain measures frequently mentioned in books, and used on particular occasions.

A Line is	$\frac{1}{12}$ Inch.
A Barleycorn	$\frac{1}{3}$ Inch.
A Palm	3 Inches.
A Hand	4 Inches.
A Span	9 Inches.
A Cubit	18 Inches.
A Pace	5 Feet.
A Fathom	6 Feet.
A Rod or Perch	$5\frac{1}{2}$ Yards.
A League	3 Miles.
A Degree	$69\frac{1}{2}$ Miles.

A Link, being one *hundredth* part of a Chain, is $7\frac{92}{100}$ inches, and a Geographical Mile is one *sixtieth* part of a Degree.

A *Barleycorn* or *Grain* of *Barley* is supposed to have been the original Element of Lineal Measure, in the same manner as a *Grain* of *Wheat* gave rise to the name of the Element of Weight.

V. TABLE OF SUPERFICIAL MEASURE.

A Square Inch is written 1 sq. in. or 1 in.

144 Square Inches are	1 Square Foot,	1 sq. ft. or 1 ft.
9 Square Feet . . .	1 Square Yard,	1 sq. yd. or 1 yd.
$30\frac{1}{4}$ Square Yards . .	1 Square Pole,	1 sq. po. or 1 po.

A square rod of $272\frac{1}{4}$ square feet is used in estimating Bricklayers' work: and a square of Flooring, Roofing, &c., is 100 square feet. See Article (190), &c.

Land Measure.

40 Poles	are	1 Rood	. . . .	1	ro.
4 Roods	. .	1 Acre	. . . .	1	ac.

Also, 1210 sq. yds. or 25000 sq. links = 1 Rood:
4840 sq. yds. or 100000 sq. links = 1 Acre.
See Article (206).

VI. Table of Solid Measure.

A Cubic Inch is written 1 cu. in. or 1 in.

1728 Cubic Inches	are	1 Cubic Foot	. 1 ft.
27 Cubic Feet	. . .	1 Cubic Yard	. 1 yd.

Also, a load of rough timber is 40 cubic feet; a load of squared timber is 50 cubic feet, and a ton of Shipping is 42 cubic feet. See Article (195), &c.

VII. Table of Measure of Capacity.

A Gill is written 1 gil.

4 Gills	are	1 Pint	1 pt.
2 Pints	. . .	1 Quart . . .	1 qt.
4 Quarts	. .	1 Gallon . .	1 gal.

By means of this measure all liquids, corn, seeds, lime, &c., are estimated according to the multiples in the following tables. See Article (209).

Ale and Beer Measure.

9 Gallons are	1 Firkin	1 fir.
2 Firkins or 18 gals	1 Kilderkin . .	1 kil.
2 Kilderkins or 36 gals . .	1 Barrel	1 bar.
$1\frac{1}{2}$ Barrels or 54 gals . . .	1 Hogshead . .	1 hhd.
2 Hogsheads	1 Butt	1 butt.
2 Butts	1 Tun	1 tun.

Wine and Spirit Measure.

10 Gallons are	1 Anker	1 ank.
18 Gallons	1 Runlet	1 run.
42 Gallons	1 Tierce	1 tier.
2 Tierces	1 Puncheon	1 pun.
63 Gallons	1 Hogshead	1 hhd.
2 Hogsheads	1 Pipe	1 pipe.
2 Pipes	1 Tun	1 tun.

Corn and Seed Measure.

2 Quarts are	1 Pottle	1 pot.
2 Pottles	1 Gallon	1 gal.
2 Gallons	1 Peck	1 pk.
4 Pecks	1 Bushel	1 bush.
2 Bushels	1 Strike	1 str.
2 Strikes or 4 Bushels	1 Coom	1 coom.
2 Cooms or 8 Bushels	1 Quarter	1 qr.
5 Quarters or 40 Bushels	1 Load	1 load.
2 Loads or 10 Quarters	1 Last	1 last.

A Sack of Flour is such a quantity as weighs 20 stone or 280 lbs., and is generally about 5 imperial Bushels. See Article (209).

Coal Measure.

4 Pecks are	1 Bushel.
3 Bushels	1 Sack.
36 Bushels	1 Chaldron.
21 Chaldrons	1 Score.

This table is of little use, as Coals are now generally sold by weight. See Article (209).

VIII. Table of Measure of Time.

A Second is written 1 sec. or 1″.

60 Seconds are	1 Minute	1 min. or 1′.
60 Minutes	1 Hour	1 hr.
24 Hours	1 Day	1 day.
7 Days	1 Week	1 wk.

On some occasions 28 days, which is nearly a *Lunar Month*, are called a Month: and a common year consists of 12 Calendar Months, or, of 12 Average Months of $30\frac{5}{12}$ days, nearly: or of 365 days, 6 hours, or of 52 weeks, 1 day, 6 hours, or of 13 months, 1 day, 6 hours, nearly, the odd day and hours being omitted in *practice*: and the numbers of days in the *Calendar* Months are usually recollected by means of the following lines.

Thirty days have September,
April, June and November:
February twenty-eight alone:
And all the rest have thirty-one;
Except in Leap-year, and then is the time,
When February's days are twenty-nine.

For an Account of the Calendar, see Article (214), &c.

IX. Table of Angular Measure.

A Second is written 1 sec. or $1''$.

60 Seconds are . .	1 Minute. . . .	1 min. or $1'$.
60 Minutes	1 Degree . . .	1 deg. or 1°.
90 Degrees	1 Right Angle.	1 rt. ang.

There are also denominations below seconds, called *thirds*, *fourths*, &c., each being one *sixtieth* part of that which precedes it; but they are generally expressed *decimally* as parts of a Second. See Article (213).

X. Table of Number, &c.

12 Units are	1 Dozen.
12 Dozen	1 Gross.
20 Units	1 Score.
24 Sheets of Paper	1 Quire.
20 Quires	1 Ream.
2 Reams	1 Bundle.
5 Bundles.	1 Bale.

A *long* hundred is 120; a *great* gross is 144 dozen: but these, and several other denominations of a similar kind are rapidly going out of use.

The Student will consult his own advantage and convenience by committing these ten tables to memory, omitting such of the Observations as may depend upon principles beyond the extent of his progress in the subject.

THE PRINCIPLES AND PRACTICE OF ARITHMETIC.

CHAPTER I.

DEFINITIONS, PRELIMINARY NOTIONS, NOTATION, NUMERATION, AND FUNDAMENTAL OPERATIONS.

ARTICLE I. DEFINITION I.

ARITHMETIC is that part of Mathematical Science which treats of the computation of magnitudes, and their relations to one another, with reference to the consideration of *how many* or *how few*.

2. DEF. 2. An *Unit*, or, as it is generally called, *Unity*, is the representation of any thing considered in its individual capacity, without regard to the parts of which it is made up, and it is the *Base* or *Element* of all arithmetical computations and comparisons.

Thus, each of the terms, *a* man, *a* house, *a* pound, &c., denotes one individual of its kind, being the same as *one* man, *one* house, *one* pound, &c., respectively; and these are the bases or elements by means of which *several* men, *several* houses, *several* pounds, &c., may be computed or compared.

3. DEF. 3. *Number* signifies a multitude or collection of two or more units, or denotes an assemblage of two or more distinct objects of the same kind.

Thus, *two* men, *three* houses, *four* pounds, &c., denote more than one individual of the same kind, the

single individuals being supposed to be repeated *twice, thrice, four times,* &c., respectively. Numbers thus viewed are termed *Whole Numbers* or *Integers,* and, for the sake of uniformity, the Unit or Unity is considered the first or least integer.

4. DEF. 4. Numbers used to express one or more individuals of *specified* kinds, as in the instances just given, are called *applicate* or *concrete* numbers; whereas *two, three, four,* &c., by themselves, not particularizing the kinds of individuals, are termed *abstract* numbers.

NOTATION.

5. DEF. 1. *Notation* is the method of expressing by means of certain symbols or characters, any proposed number or quantity arithmetically considered.

6. DEF. 2. The *Symbol* or *Representative* of unit or unity, is 1; but instead of any other number being expressed by an assemblage or multitude of units placed together, which would soon become embarrassing, other characters or symbols have been invented, by means of which every number, however small or great, may be expressed; and instead of a different symbol being adopted for every different number, which would soon become equally inconvenient, all numbers are expressed by means of the following *ten* symbols, or as they are usually termed, *Figures,* and sometimes *Digits,* which have their names respectively annexed:

1,	2,	3,	4,	5,	6,	7,	8,	9,	0:
one,	two,	three,	four,	five,	six,	seven,	eight,	nine,	zero:

the first *nine* of which are all defined by their names; and the last, which is variously denominated *Nought, Cipher,* or *Zero,* when standing by itself has no signification, or at most, denotes the absence of number, and is to be regarded merely as an *auxiliary* digit, for the purposes hereafter to be explained.

7. DEF. 3. Whenever any figure is placed on the right of the same or any other, it has, by universal agreement, the effect of increasing the value of the last mentioned figure *tenfold,* at the same time that it retains its own value.

Thus, beginning with the auxiliary digit 0, we have the following numbers and their representations;

10,	11,	12,	13,	14,	&c:
ten,	eleven,	twelve,	thirteen,	fourteen,	&c:

20,	21,	22,	&c:
twenty,	twenty-one,	twenty-two,	&c:

and it is obvious that by means of two figures, this kind of notation may be continued till we arrive at *ninety-nine,* whose symbol will be 99.

8. Def. 4. Beyond this number, the use of *two,* either the same or different figures will not enable us to go, but a repetition of the contrivance explained in the last article, will by means of more figures supply the defect.

Thus, beginning again with the auxiliary digit 0, and supposing the effect of any figure's being placed on the right of symbols formed as above, to be to increase all their values *tenfold,* we shall have

100,	101,	102,	&c:
one hundred,	one hundred and one,	one hundred and two,	&c:

so likewise of succeeding numbers; thus,

345,	586:
three hundred and forty-five,	five hundred and eighty six:

and again, 999 will be *nine hundred and ninety-nine,* which is the largest number capable of being expressed by *three* figures. Here the first figure on the right hand is said to occupy the *units' place,* the second the place of *tens,* and the third that of *hundreds.*

Of the auxiliary digit 0, the sole use is in the effect specified in the last two articles, and all figures to the right of it will therefore be unaffected by it.

9. Def. 5. Before we proceed further, we may observe that it is usual, in estimating numerical magnitudes, to proceed in order from *hundreds* to *thousands, tens of thousands, hundreds of thousands, millions, tens of millions,* and *hundreds of millions,* in precisely the same manner as we have done from *units* to *tens,* and from *tens* to *hundreds,* in the preceding articles.

10. Def. 6. By a generalization of the principle adopted in article (7), it is assumed that " any figure placed on the right of one or more others, has the effect of increasing every one of them tenfold, without being affected in its own value;" and we are thus enabled to express with facility all numbers whatsoever.
Thus,

1000 will represent One thousand.

5493 will represent Five thousand, four hundred and ninety-three.

23456 will represent Twenty-three thousand, four hundred and fifty-six.

729054 will represent Seven hundred and twenty-nine thousand and fifty-four.

1803205 will represent One million, eight hundred and three thousand, two hundred and five.

32754081 will represent Thirty-two millions, seven hundred and fifty-four thousand and eighty-one.

473025004 will represent Four hundred and seventy-three millions, twenty-five thousand and four.

And similarly, for larger numbers.

11. DEF. 7. If the first three figures beginning from the right hand be denominated so many *units*, tens of *units* and hundreds of *units*, it follows that the next three figures taken the same way will be *thousands*, tens of *thousands* and hundreds of *thousands*: the next three in order, will be *millions*, tens of *millions* and hundreds of *millions*, and so on; and hence to express any number proposed, we have only to consider in which of these divisions each part of it ought to be found, observing that *three* figures from the right must be taken to make each division *complete* before we proceed to the next.

Ex. 1. Express by means of figures; *Thirty-five thousand, eight hundred and nineteen.*

Here, eight hundred and nineteen belongs to the first division on the right, and is written 819:

also, thirty-five thousand must be found in the second division from the right, and is 35:

whence the proposed number will be expressed in figures by

3 5 8 1 9.

Ex. 2. Write down in figures the number; *Five millions, twenty-five thousand, six hundred and seven.*

In this case, the first division on the right will be 607; the second will be 025, the digit 0 being affixed to the left of the others without altering their values, to make up the required number of *three*, and the third is 5; so that the expression required will be

5 0 2 5 6 0 7.

Ex. 3. Express by figures the following number;

Five hundred and seventy millions, two hundred and six thousand and fifty-four.

Here, the first division is 054, the 0 altering only the values of the figures in the subsequent divisions: the second division is 206, and the third is 570: whence the number proposed is correctly expressed by

5 7 0 2 0 6 0 5 4.

12. Examples of the kind just given, might easily be multiplied, but the method of notation can never present any difficulty, provided it be carefully remembered that every division of figures as we proceed from the right hand towards the left must be *completed* as far as it is possible; and indeed by a little practice, we shall soon be enabled to write down any proposed number by beginning at the left hand.

Ex. To write down *Six hundred and thirteen millions, five hundred and nineteen,* we observe that the division of millions will be 613: that of thousands 000, and that of units 519: and the number expressed by the arithmetical symbols is

6 1 3 0 0 0 5 1 9.

13. A facility in expressing arithmetically, any numerical magnitude that may be presented to his notice, being of the greatest importance to the student, the following additional examples for practice are subjoined.

(1) Five hundred and ninety-eight.

(2) Seven thousand, eight hundred and four.

(3) Eighty-nine thousand and sixty-three.

(4) Six hundred and three thousand, two hundred and forty.

(5) Nine millions, forty-three thousand, six hundred and two.

(6) Forty-five millions, three hundred and eighty-seven thousand and twenty-five.

(7) Three hundred and forty-nine millions, four thousand and sixty-five.

(8) One hundred millions, ten thousand and one.

(9) Eight hundred and forty-two millions, two hundred and forty-eight thousand, four hundred and eighty-four.

(10) Nine hundred and nine millions, nine thousand and ninety-nine.

14. As far as practical utility is concerned, we shall seldom or never have occasion to express by figures, numbers exceeding *Hundreds of Millions;* but the system of Notation admits of being extended so as to represent any number whatever.

Thus, instead of supposing that each division consists of *three* figures, if we include *six* figures as far as we can in each division, the first may be regarded as so many hundreds of thousands of *Units;* the next as so many hundreds of thousands of *Millions;* the next as so many hundreds of thousands of what are called *Billions,* and the succeeding divisions, of so many hundreds of thousands of what are termed *Trillions, Quadrillions,* &c.

Ex. To represent *Ten thousand millions* by figures; for the first division we have, according to this view, 000000, and for the second 10000, so that the representation required is

1 0,0 0 0,0 0 0,0 0 0.

15. It will readily be observed, from what has already been said, that each of the nine figures or digits,

1, 2, 3, 4, 5, 6, 7, 8, 9,

has an absolute value of itself, whereas the auxiliary digit 0 has no such value; and on this account the former are sometimes termed *significant* figures, in contradistinction to the last. It will moreover have occurred to the reader, that every one of these significant digits, in addition to its *absolute* value, which is fixed and certain, possesses also a *local* value dependent upon the situation in which it is placed; thus, in the expression of the number

Four thousand, three hundred and twenty-one,

which will be

4 3 2 1,

the 1 in the first place on the right hand, retains its absolute value; the second figure 2, in its situation denotes two *tens* or *twenty;* the third is three *hundred,* and the fourth is four *thousand;* so that the local values of 2, 3, and 4, are respectively, *ten* times, a *hundred* times and a *thousand* times, as great as their absolute values: and it is the circumstance of assigning to each of the significant figures a local as well as an absolute value, which confers

upon the system, the immense powers it possesses of being adequate to the representation of any number, however great, as already shewn.

NUMERATION.

16. DEF. *Numeration* is the art of reading or estimating the value of any number, expressed by means of the numeral characters in whatever manner combined or repeated, and is therefore the reverse of Notation.

17. From the circumstance of every figure possessing a local as well as an absolute value, it follows that the value of each must be estimated by the place which it occupies: hence, therefore, a figure standing by itself expresses so many *units;* a figure in the second place from the right denotes so many *tens;* a figure in the third place, so many *hundreds,* and so on, according to articles (10) and (11): consequently, if we suppose any numerical expression to be divided into portions, each consisting of three figures as far as they go, the figures of the division on the right will be *units,* and tens and hundreds of *units;* those of the next division will be units, tens and hundreds of *thousands;* those of the third will be units, tens and hundreds of *millions,* and so on. Thus,

25 is Twenty-five.

304 is Three hundred and four.

5287 is Five thousand, two hundred and eighty-seven.

60539 is Sixty thousand, five hundred and thirty-nine.

207385 is Two hundred and seven thousand, three hundred and eighty-five.

1739204 is One million, seven hundred and thirty-nine thousand, two hundred and four.

35024376 is Thirty-five millions, twenty-four thousand, three hundred and seventy six.

275008005 is Two hundred and seventy-five millions, eight thousand and five.

In every one of these instances we conceive the expression to be separated into portions of three figures each as far as they go, beginning at the right hand: as in 275008005, we observe that 005 is the first portion, 008 the second, and the third portion is 275, each consisting of three figures: that is, 275 denotes two hundred and seventy-five millions, 008 eight thousand and 005 five units, and the expression will be read as above.

18. The substance of the last article will be rendered still more clear by means of the following scheme, which is called the Numeration Table:

&c. &c.	Hundreds of Millions.	Tens of Millions.	Millions.	Hundreds of Thousands.	Tens of Thousands.	Thousands.	Hundreds.	Tens.	Units.
	9	8	7	6	5	4	3	2	1
		9	8	7	6	5	4	3	2
			9	8	7	6	5	4	3
				9	8	7	6	5	4
					9	8	7	6	5
						9	8	7	6
							9	8	7
								9	8
									9

wherein the local value of every figure in each of the horizontal rows is pointed out by the name written upwards at the top of the whole: thus, in the third horizontal line from the bottom, the figures will be read *Nine hundred and eighty-seven;* and in the second line from the top, *Ninety-eight millions, seven hundred and sixty-five thousand, four hundred and thirty-two.*

19. For practice, the student is advised to write down in words at length, the following numerical expressions.

(1) 4320;
(2) 87054;
(3) 903756;
(4) 2714325;
(5) 8047328;
(6) 12870045;
(7) 20084216;
(8) 79030284;
(9) 321408653;
(10) 408076032;
(11) 314159265;
(12) 571268405.

20. The principles of Notation, or the *expressing* of any number by means of the ten numeral characters, and those of Numeration, or the *reading* of numerical magnitudes so expressed, being once established and understood, we proceed to the consideration of the four funda-

mental Arithmetical Operations that can be performed upon numbers, which are those of *Addition, Subtraction, Multiplication* and *Division,* each of which will be defined, explained and exemplified in order.

I. ADDITION.

21. Def. *Addition* is the first of the fundamental operations of Arithmetic, and consists in finding a number equal to the aggregate of two or more numbers taken together, and this number is called their *Sum.*

Ex. 1. To find the sum of the simple numbers 2, 5 and 9; we see that *two* units and *five* units taken together make *seven* units, and this with *nine* units more, will manifestly amount to *sixteen* units, which is written 16.

The operation may stand as follows:

$$\begin{array}{lrl} & 2 & \\ & 5 & \\ & 9 & \\ \hline \text{therefore} & 16 & \text{is the sum.} \end{array}$$

Ex. 2. Add together the numbers expressed by 254, 893 and 487.

Here it would be absurd to collect *immediately* into one sum, numbers of different local values, as for instance, to say that three *units* and five *tens* amount together to either eight *units* or eight *tens,* and we therefore place the numbers to be added together in such a form that each of the figures of the same denomination may be in the same vertical line, as on the left of the page:

Common Form.	*Explanation of Operation.*				
2 5 4	2 0 0	and	5 0	and	4
8 9 3	8 0 0	...	9 0	...	3
4 8 7	4 0 0	...	8 0	...	7
2 1	1 4 0 0		2 2 0		1 4
1 6 3 4 the sum.	2 0 0		1 0		
	1 6 0 0		2 3 0		

and then, as is seen in the operation on the right, we have first added the units together and thus have 14 units, or 1 ten and 4 units: we have next found the sum of the tens to be 22, which with the 1 ten before obtained amount to 23 tens, or 2 hundreds and 3 tens; and lastly, we have by the same kind of process ob-

tained 14 hundreds, which together with the 2 hundreds last found make 16 hundreds, or 1 thousand and 6 hundreds: whence the entire sum is 1 thousand, 6 hundreds, 3 tens and 4 units, or 1634.

The reasoning here used is thus applied to the figures on the left of the page: the numbers of tens and hundreds found by adding the vertical columns of units and tens are annexed, or *carried* to the columns of tens and hundreds respectively, and they are here put down under them just above the horizontal line; but in practice they are generally omitted altogether by *mentally* adding them to the lowest figures of the next vertical rows, and then proceeding as before.

22. To effect the operation of Addition, as appears from the two instances just considered, it is therefore merely necessary to know from *memory* or by *practice*, the sums of every two numbers expressed by single figures, and the reasoning above employed leads to a general conclusion which is comprised in the following Rule.

Rule for performing Addition.

Place the numbers under one another in such a manner that units may stand under units, tens under tens, hundreds under hundreds, and so on, and draw a line below all the horizontal rows of figures: then add up the figures in the first vertical row on the right hand, find the numbers of *tens* and *units* in their sum, and put down the number of *units*, whether it be zero or any of the nine other digits: *carry* as many *units* as there are *tens* thus found to the next vertical row, and add them up as before, observing the numbers of *tens* and *units* contained in the sum: place the number of *units* under the row added, and carry the number of *tens* to the next; proceed in the same manner till the last row is added, when put down both the numbers of *tens* and *units*, as there are no more figures of higher denominations.

23. To ascertain whether the operation is correctly performed, various expedients might be resorted to; as for instance, that of adding the numbers *downwards* instead of *upwards*, which, because the *same* set of numbers cannot have two *different* sums, must give the same result as before: but the only one, with this exception, which does not involve principles hereafter to be explained,

seems to be that of omitting any one of the horizontal rows of figures in a *second* operation, and afterwards adding it to the result thus obtained, as in the following example:

Addition.	*Proof.*
9358	4162
4162	8920
8920	6328
6328	19410
28768	9358
	28768

where 28768 is the sum: and omitting the first horizontal row of figures, we find the sum of the rest to be 19410, and to this the row 9358 omitted being now added produces 28768 the entire sum as before: whence we infer with some degree of probability, that the addition is correct: and this probability may be still further increased by repeating the operation, with the omission of any other horizontal row of figures *different* from the one already left out.

24. We will now place before the student a few examples for practice, some of which are properly arranged for the immediate performance of the operation, and the rest are to be first adapted for that purpose.

(1)	(2)	(3)	(4)
90	347	7153	29051
45	238	2857	73826
73	410	4105	57295

(5)	(6)	(7)	(8)
84	293	4028	53296
7	75	354	109
29	409	95	5875
13	3	2076	24658

(9)	(10)	(11)	(12)
73	235	736	25385
24	97	400	90624
9	958	4159	87653
251	64	47	40706
48	186	7204	97341

(13) Add together 432, 8076, 458 and 5431.

(14) Add together 72853, 27621, 45760, 820547 and 71425.

(15) Add together 205087, 32471, 29185, 1475 and 273.

(16) Find the sum of 72638594, 27836, 7805, 5271 and 1468357: and prove it to be correct by the omission of each horizontal row in succession.

(17) Find the sum of *Twenty-five millions and four; Forty-seven thousand, two hundred and nine; Three hundred millions, ten thousand and one; Sixty-five thousand and eighty-seven, and Five millions and fifty*: write it down in words; and apply the ordinary proof of its being correct.

25. It is usual, in many of the applications of Arithmetic, to express the operation of *Addition* by means of *signs* invented for the purpose: thus, the sum of 4 and 5 is expressed in the form,

$$4 + 5 = 9,$$

wherein the sign + between 4 and 5 denotes the addition of the latter number to the former, and is read *plus* or *more by*; and the sign = between 5 and 9 expresses the result of such addition to be 9, or the *equality* between the sum of the numbers 4 and 5 and the number 9: so that the arithmetical *expression*

$$4 + 5 = 9,$$

is read

4 *plus* 5 *equals* 9.

Similarly, $2 + 3 + 7 = 12$, shews the sum of the three digits 2, 3, 7, to be 12: and the same observation may be made, whatever be the numbers to be added, as in Ex. 2, of Article (21), we have $254 + 893 + 487 = 1634$, *expressive* of the operation there *performed*.

II. SUBTRACTION.

26. Def. *Subtraction* is the second of the fundamental operations of Arithmetic, and consists in finding a number equal to the excess of one number above another, and this excess is styled the *Difference* or *Remainder*. The greater of the numbers is sometimes called the *Minuend*, and the less the *Subtrahend*.

Ex. 1. Let it be required to find the difference of 7 and 2.

Here it is evident that 7 units being equal to 2 units and 5 units taken together, if we withdraw the former, we shall have 5 units for the difference.

The numbers and operation are usually expressed as below:

$$\begin{array}{rcl} & 7 & \\ & 2 & \\ \hline \text{therefore} & 5 & \text{is the difference.} \\ \hline \end{array}$$

Ex. 2. To subtract the number 19 from the number 37, we place the figures as in the last example, and have

Common Form.	*Explanation of Operation.*
37	20 and 17
19	10 ... 9
18	10 ... 8

where the figure in the units' place of the upper line being less than that in the lower, it is manifestly impossible to substract the lower from the upper: but by considering, as on the right of the page, the 7 as 17 by taking one of the units from the 3, we find the excess of 17 above 9 to be 8, which is put in the units' place of the remainder, and then we have to take away 1 from 2 instead of 3, in consequence of having regarded the 7 as 17: hence the remainder in the tens' place will be 1, and the difference of the two numbers is therefore 18, as exhibited on the left.

When the figure in the lower line is greater than that in the upper, we have *borrowed* ten units of the *next* denomination; but the same result is obtained whether we suppose 1 to be subtracted from the upper line, or added to the lower, as the remainder will evidently be the same on both suppositions. In practice we add ten units of any denomination to both the quantities concerned; to the upper as *ten* of that denomination, and to the lower as *one* of the next superior denomination, and by this contrivance the remainder is clearly unaffected.

27. From what has been done in these examples, it will appear to be necessary to *recollect* for this and other purposes, the differences of every two numbers less than 20: and the reasoning here used being applicable to

all other instances, the result of it may be embodied in the following rule.

Rule for performing Subtraction.

Place the less number under the greater, so that units may stand under units, tens under tens, and so on, as before; begin at the units' place and subtract each figure in the lower line from that in the upper, taken by itself, or increased by 10, according as it is greater or less than the said figure in the lower line, and put down the remainder, observing that whenever *ten* units of any denomination have been *borrowed*, or added to the upper line, *one* unit must be added to the next denomination in the lower line.

28. The operation of Subtraction being the reverse of that of Addition, it follows, that if we add together the remainder and the less of the numbers proposed, the sum thus obtained ought to be equal to the greater; and the operation of subtraction may be presumed generally to be correct when this is the case. Thus, in the following example:

Subtraction.		*Proof.*	
9628	= Minuend:	6759	= Subtrahend:
6759	= Subtrahend:	2869	= Remainder:
2869	= Remainder:	9628	= Minuend:

where the last result is the same as the greater of the numbers proposed, as it ought to be; and thence we infer that the required operation has been correctly performed.

29. The following examples, partly arranged, and partly not, are intended for practice in performing the operation of Subtraction, and also in applying the method of proof.

	(1)	(2)	(3)	(4)
	14	79	428	7046
	8	45	274	807

	(5)	(6)	(7)
	62831	542657	204087
	48072	214958	76498

(8) What is the excess of 12795 above 8096?

(9) From 9261374 take 2548298.

(10) Find the difference of 20470932 and 80476325.

(11) How much greater is 12785462 than 1842567?

(12) Required the excess of *Three hundred and five millions, two hundred and four*, above *Seventy-five thousand, three hundred and eighty-six.*

30. The operation of Subtraction, in like manner as that of Addition, is indicated or expressed by means of the sign $-$, which is read *minus* or *less by;* thus, the excess of 7 above 3, will be expressed in the form,

$$7 - 3 = 4,$$

which is read

7 minus 3 equals 4:

where the sign $-$ between 7 and 3 denotes the subtraction of the latter from the former, and the sign $=$ between 3 and 4 shews the *equality* of the excess to 4.

III. MULTIPLICATION.

31. Def. *Multiplication* is the third of the fundamental operations of Arithmetic, and consists in finding the amount of a number, when repeated any number of times, and this amount is termed the *Product.* The former of these numbers is called the *Multiplicand,* and the latter the *Multiplier.*

Ex. 1. To multiply the numbers 7 and 42 by the numbers 4 and 5 respectively, being to find the sums arising from the numbers 7 and 42 *four* and *five* times repeated, we may determine the products as underneath;

$$\begin{array}{r}7\\7\\7\\\underline{7}\\28\end{array}\qquad\qquad\begin{array}{r}42\\42\\42\\42\\\underline{42}\\210\end{array}$$

but the operations are expressed more briefly, as follows:

$$\begin{array}{r}7\\\underline{4}\\28\end{array}\qquad\qquad\begin{array}{r}42\\\underline{5}\\210\end{array}$$

Ex. 2. Find the products arising from the multiplication of 256 by 10, 11 and 12 respectively.

By article (10) we know that 256 will become *ten* times as great by merely affixing to the right of it the auxiliary digit 0, which has the effect of increasing the value of every figure tenfold, and thus we have the following operation:

$$\begin{array}{r l} 2\;5\;6 & \text{the multiplicand:} \\ 1\;0 & \text{the multiplier:} \\ \hline 2\;5\;6\;0 & \text{the product.} \\ \hline \end{array}$$

To multiply 256 by 11, we have only to consider that 11 being equal to 1 and 10 taken together, the required product will be equal to the sum of 256 taken *once* and *ten* times: thus,

$$\begin{array}{r l} 2\;5\;6 & \\ 1\;1 & \\ \hline 2\;5\;6 & = 256 \text{ taken } \textit{once}, \\ 2\;5\;6\;0 & = 256 \text{ taken } \textit{ten} \text{ times}, \\ \hline 2\;8\;1\;6 & = 256 \text{ taken } \textit{eleven} \text{ times:} \\ \hline \end{array}$$

that is, 2816 is the product of 256 by 11, and the omission of the 0 at the right of the fourth line in the operation, can cause no inconvenience, as the *places* of the succeeding figures adequately determine their values.

To find the product of 256 by 12, it follows as above that 256 must be taken *twice* and *ten* times together, and thus we have the following operation:

$$\begin{array}{r l} 2\;5\;6 & \\ 1\;2 & \\ \hline \left.\begin{array}{r} 2\;5\;6 \\ 2\;5\;6 \end{array}\right\} & = 256 \text{ taken } \textit{twice}, \\ 2\;5\;6\;0 & = 256 \text{ taken } \textit{ten} \text{ times}, \\ \hline 3\;0\;7\;2 & = 256 \text{ taken } \textit{twelve} \text{ times:} \\ \hline \end{array}$$

whence the product of 256 by 12 is 3072, the observation above made holding good with respect to the omission of the 0 at the end of the fifth line of the operation.

32. From the mode in which the results of the examples just given have been obtained, it is manifest that the operation of Multiplication is merely a compendious method of performing the addition of two or more *equal* numbers: and the following scheme, which is termed the *Multiplication Table,* presents at one view the product arising from the multiplication of any two numbers not exceeding 12; and though the products of the nine digits form the basis of those of all numbers whatever, it is here extended for the sake of *practical* convenience, and should be carefully committed to memory.

THE MULTIPLICATION TABLE.

1	2	3	4	5	6	7	8	9	10	11	12
2	4	6	8	10	12	14	16	18	20	22	24
3	6	9	12	15	18	21	24	27	30	33	36
4	8	12	16	20	24	28	32	36	40	44	48
5	10	15	20	25	30	35	40	45	50	55	60
6	12	18	24	30	36	42	48	54	60	66	72
7	14	21	28	35	42	49	56	63	70	77	84
8	16	24	32	40	48	56	64	72	80	88	96
9	18	27	36	45	54	63	72	81	90	99	108
10	20	30	40	50	60	70	80	90	100	110	120
11	22	33	44	55	66	77	88	99	110	121	132
12	24	36	48	60	72	84	96	108	120	132	144

In this, the first horizontal line consists of the first twelve numbers in order: the second consists of the products of the same numbers when multiplied by 2: the third con-

tains their products when multiplied by 3: the fourth when multiplied by 4, and so on: and the table is usually repeated in the following manner:

thus, to make use of the second line of figures, we say

twice 1 are 2,	twice 5 are 10,	twice 9 are 18,
twice 2 are 4,	twice 6 are 12,	twice 10 are 20,
twice 3 are 6,	twice 7 are 14,	twice 11 are 22,
twice 4 are 8,	twice 8 are 16,	twice 12 are 24;

and so on: but the utility and importance of this table will be fully evinced in the progress of the work, which almost entirely depends upon it.

Ex. 1. Let it be required to multiply 854 by the single figure 6: then, since the product of 854 by 6 is evidently equal to the sum of the products of all its parts, namely, 800 and 50 and 4, by 6, we have the following operation:

$$
\begin{array}{rl}
854 & \\
6 & \\
\hline
24 & = \text{product of } 4 \text{ by } 6: \\
300 & = \text{product of } 50 \text{ by } 6: \\
4800 & = \text{product of } 800 \text{ by } 6: \\
\hline
5124 & = \text{product of } 854 \text{ by } 6: \\
\hline
\end{array}
$$

but in practice, we *mentally* combine into one sum, the figures of all these products as they arise: thus, first multiplying 4 by 6, we find the product to be 24 by the table, and having placed the 4 *units* under those of the quantity proposed, we carry the 2 *tens* to the product of 5 by 6, which is here 30 *tens*, and thus obtain 32 tens, whereof the 2 being put under the *tens'* place, and the 3 being carried to the product of 8 by 6, or to 48 *hundreds*, the entire number of hundreds is 51, and the whole product is 5124: and it is evident that if the multiplicand comprise more figures, the process has only to be continued.

Ex. 2. Multiply 486 by 357.

Here, proceeding with each of the figures 7, 5 and 3, according to the principle of the last example, we have

$$\begin{array}{r@{\;}l}
486 & \\
357 & \\
\hline
3402 & = \text{product of 486 by 7:} \\
2430 & = \text{product of 486 by 50:} \\
1458 & = \text{product of 486 by 300:} \\
\hline
173502 & = \text{product of 486 by 357:}
\end{array}$$

and in this the *situations* of the figures in the fourth and fifth lines are sufficient to render them equivalent to the products of 486 by 50 and 300 respectively, without supplying the places of units, and of units and tens, with the auxiliary digit 0.

If one or more of the figures of the multiplier be 0, it is evident that the corresponding *partial* product will be 0, and the lines may be entirely omitted after placing down each 0 *once*, to give its proper value to the product arising from the next figure.

33. The reasoning here employed being independent of the particular examples made use of to illustrate it, we are enabled to lay down a rule in the following form.

Rule for performing Multiplication.

Place the multiplier under the multiplicand, as in the preceding operations, and draw a line under the whole: multiply every figure in the multiplicand by the figure in the units' place of the multiplier, observing to carry to the next product the number of *tens* in that arising from the multiplication of any of the digits in the multiplicand, and to place down the *units* under the figure multiplied, till the last product is obtained, which place down in full: proceed in the same manner with the figure of the multiplier in the tens' place, the figure on the right of this product being placed under the said figure; then with the figures in the succeeding places; add all these products together, and the sum will be the entire product of the numbers proposed.

34. Without involving higher principles than have been explained, we may observe, that if the multiplicand and multiplier change places, the product must be the same as before, otherwise the same numbers would have more products than one; and if the products be the

same, we have some proof that the operation has been correctly performed in each case: thus, taking the following example, we have

Multiplication.	*Proof.*
875	423
423	875
2625	2115
1750	2961
3500	3384
370125	370125

where we perceive that the products of the two operations are the same; and this circumstance is a strong proof that both operations are correct.

Abbreviations, &c. of Multiplication.

35. It frequently happens that deviations from the ordinary process of Multiplication may be adopted, in order to shorten or facilitate the operation, as will be exemplified in the following instances.

Ex. 1. Multiply 257 by 6400, and 790 by 8300.

Here, omitting the ciphers on the right of the multiplicand and multiplier, or *supposing* them to be omitted, we have

257	790
6400	8300
1028	237
1542	632
1644800	6557000

the ciphers being annexed to the right of the products obtained in the ordinary way, to give the other figures their proper local values.

Ex. 2. Required the product of 537 by 63.

Here 63 being equal to the product of 7 and 9, it follows that 7 times any number 9 times repeated, is the same as 63 times that number: whence we have

```
  5 3 7
      7  times 9 = 63
 ------
3 7 5 9
      9
-------
3 3 8 3 1 = the product.
---------
```

Similarly, to multiply 476 by 47, we have

```
    4 7 6   |      4 7 6
        9   |          8
  -------   |    -------
  4 2 8 4   |    3 8 0 8
        5   |          6
---------   |  ---------
2 1 4 2 0   |  2 2 8 4 8
    9 5 2   |      4 7 6
---------   |  ---------
2 2 3 7 2   |  2 2 3 7 2
---------   |  ---------
```

in the former of which *twice* 476 is *added*, and in the latter *once* 476 is *subtracted*, to complete the multiplier 47.

36. We have as yet considered the multiplication of two numbers only; but it is evident that the same mode of reasoning and similar operations may be used to find the product of more than two, which is usually called the *Continued Product* of so many *Factors*.

Ex. To find the product arising from the continued multiplication of the numbers 3, 5 and 47, we have first to

```
          multiply  3
                by  5
                   ---
and therefore      15 is the product:
                   ---
again, we multiply 15
                by 47
                   ---
                   105
                   60
                   ---
```

and the product is 705, which is called the continued product of the three factors 3, 5 and 47.

37. The following examples without their answers are intended for practice, and in which all the principles both of operation and proof hitherto explained, are called into use.

(1)	284	(2)	1475	(3)	2867
	2		3		4
(4)	78543	(5)	41087	(6)	942763
	5		6		7
(7)	8536274	(8)	3216795	(9)	1468725
	8		9		11
(10)	6283195	(11)	21584	(12)	39265
	12		17		39
(13)	921846	(14)	827941	(15)	5086927
	158		376		495
(16)	279420	(17)	254037	(18)	4785328
	7350		2980		7802

(19) Multiply 123456789 by each of the numbers 2, 3, 4, 5, 6, 7, 8 and 9.

(20) Find the respective products of 47691 and 27: of 28573 and 35: of 716281 and 48: of 129385 and 66: of 138476 and 81: of 480765 and 97, and of 8241763 and 123.

(21) Required the continued products of 4, 7 and 25: of 13, 15 and 17: and also of 35, 29, 43 and 87.

38. The operation of Multiplication is expressed by the sign × which is read *into*, or *multiplied by*: thus,

$$5 \times 7 = 35$$

denotes that the result of the multiplication of 5 by 7 is 35: so, again,

$$4 \times 5 \times 13 = 260$$

expresses the continued product of the numbers 4, 5 and 13: and employing the signs of the preceding operations of Addition and Subtraction, we have

$$(8 + 3) \times (7 - 2) = 55,$$

expressive of the product of the two numbers formed by the *sum* of 8 and 3, and the *difference* of 7 and 2 respectively, and which may be more briefly written

$$11 \times 5 = 55.$$

IV. DIVISION.

39. DEF. *Division* is the last of the fundamental operations of Arithmetic, and consists in finding how many times one number is contained in another, and the number of such times is termed the *Quotient*. The former of these numbers is called the *Divisor*, and the latter the *Dividend*.

Ex. 1. To divide 6 by 2, and 219 by 52 respectively, we must obviously take the latter numbers from the former in each case, as often as we are able, according to the principle of Subtraction before explained: thus,

$$\begin{array}{r|r}
 & 219 \\
 & \underline{52} \\
6 & 167 \\
\underline{2} & \underline{52} \\
4 & 115 \\
\underline{2} & \underline{52} \\
2 & 63 \\
\underline{2} & \underline{52} \\
\underline{0} & \underline{11}
\end{array}$$

so that after *three* subtractions in the former case, there is *no* remainder, whereas in the latter, *four* such operations being performed leave a remainder 11: that is, 2 is con-

tained in 6, *three* times exactly, but 219 divided by 52 gives 4 for the quotient, with 11 remaining over and above: and the processes are usually made to take the following forms:

2) 6	52) 219 (4
3	208
	11

From these instances of *Short* and *Long* division, it evidently follows that Division is the reverse of Multiplication: and hence by a reversed process, the Multiplication Table must furnish the means of obtaining the quotient.

Ex. 2. If we multiply 349 by 215, the product is 75035: and therefore by the last example, the quotient of 75035 when divided by 349 must be found to be 215, by reversing the operation as follows:

```
349 ) 7 5 0 3 5 ( 2 1 5
      6 9 8        = product of 349 by 2 denoting 200 units:
      -----
        5 2 3
        3 4 9      = product of 349 by 1 denoting 10 units:
        -------
        1 7 4 5
        1 7 4 5    = product of 349 by 5 denoting 5 units:
        -------
```

and in this, the first figure 2 in the quotient is obtained by enquiring how often 3 is contained in 7, or 34 in 75: then after multiplying 349 by 2, which, from the places of the figures, represents 2 *hundreds*, and subtracting the product, which is 698, from 750, we have a remainder 52: to this the next figure 3 of the dividend being annexed, we seek how often 3 is contained in 5, or 34 in 52, and this quotient being 1, 1 *ten* is annexed to the 2 *hundreds* already obtained: multiplying and subtracting as before, we bring down the last figure 5 of the dividend, and find the corresponding quotient to be 5 *units* exactly; and the operation is then completed, leaving no remainder, as it ought.

Here the dividend has virtually been broken up into parts each exactly divisible by 349, which will appear more clearly, by supplying the auxiliary digits throughout the process, in the form of *Long Division*: thus,

Divisor.	*Dividend.*	*Quotient.*
349)	69800 + 3490 + 1745 (	200 + 10 + 5 :
	69800	
	+ 3490	
	3490	
	+ 1745	
	1745	

or, in the form of *Short Division,* below:

Divisor.	*Dividend.*
349)	69800 + 3490 + 1745
	200 + 10 + 5

40. The principles of the reasoning employed in these examples being the same for all cases, may be embodied in the following general rule.

Rule for performing Division.

Place the divisor and dividend in the same horizontal line one after the other, separated by a curved line; and on the right of the dividend draw another line of the same kind: enquire how often the first one or two figures on the left hand of the divisor is contained in the first one or more of those of the dividend, and place the result on the right as the first figure of the quotient: and the product arising from the multiplication of the divisor by this figure being subtracted from the dividend, bring down and annex to the remainder the next figure of the dividend, and let the same kind of operation be repeated till every figure of the dividend is thus disposed of, and both the quotient, and the remainder, if any, will be ascertained.

If the divisor do not exceed 12, these operations may be performed *mentally,* and the quotient and remainder placed in a line immediately under the dividend, as in the last part of the preceding article.

41. Since the quotient is the result arising from the division of the dividend by the divisor, it follows that the dividend must be the product arising from the multiplication of the divisor by the quotient, or of the quotient by the divisor: also, if there be any remainder, it must evidently be added to this product to produce the true dividend, since the whole is equal to the sum of all its

parts; and hence we have a method of proving whether the division has been correctly performed.

Ex. Let it be required to find the quotient and remainder when 275487 is divided by 736; and to apply the method of proof.

Division.

```
736)275487(374
    2208
    ----
     5468
     5152
     ----
      3167
      2944
      ----
       223
```

Proof.

```
   374
   736
  ----
  2244
 1122
2618
------
275264
   223
------
275487
```

Abbreviations, &c. of Division.

42. The operation of Division may, in particular cases, be made to comprise fewer figures, or to take up less room, by such considerations as are used in the following examples.

Ex. 1. If we wish to divide 20573290 by 34500, we proceed as follows:

```
Divisor.     Dividend.      Quotient.
345,00)205732,90(596;
       1725
       ----
        3323
        3105
        ----
         2182
         2070
         ----
          11290 remainder;
```

where, after the two *ciphers* in the divisor and the two *figures* 90 in the dividend are *cut off*, the operation is effected by the ordinary method, the said two figures of the dividend being affixed to the remainder at last, inasmuch as 112 from the places of the figures is equivalent to 11200.

Ex. 2. Let it be required to divide 792415 by 72.

Here, since 72 is the product of 8 and 9, it is obvious, from Ex. (2), of Article (35), that the quotient may be

obtained from successive divisions by 8 and 9, arranged as follows:

$$72\left\{\begin{array}{l} 8\,)\,\underline{7\;9\;2\;4\;1\;5} \\ 9\,)\;\;\underline{9\;9\;0\;5\;1},\ 7 \text{ first remainder:} \\ \qquad\;\; 1\;1\;0\;0\;5,\ 6 \text{ second remainder:} \end{array}\right.$$

and we have now only to deduce the true remainder from the two remainders just found. The dividend at first being so many *units*, the first remainder 7 must evidently be *units;* but the second dividend being the result of the division by 8, must clearly be regarded as so many times 8, and the second remainder will therefore be 6 times 8, or 48 *units:* whence, if to this the 7 *units*, already found, be added, the true remainder will be

$$6 \times 8 + 7 = 55:$$

and we may lay down a rule in the following words.

In dividing by *two* numbers, instead of *one* equal to their product, the *true remainder* is equal to the product of the *last remainder* and the *first divisor*, together with the *first remainder*.

43. The following examples without their answers, are intended for the student's exercise on the rules and remarks made in this section.

(1) 2) 348 (2) 3) 4596 (3) 4) 276284

(4) 5) 84375 (5) 6) 53844 (6) 7) 536074

(7) 8) 95832417 (8) 9) 7163253651

(9) 10) 3158367 (10) 11) 1234567890

(11) 12) 9876543 (12) 23) 14415724 6(

(13) 37) 47073256((14) 549) 48310567(

(15) 7038) 140167329((16) 7900) 25413286(

(17) 5730) 8327970((18) 1480) 64157600(

(19) Find the respective quotients of 76294 by 32: of 729518 by 49: of 8015473 by 66, and of 4050873 by 121; and prove the correctness of the operations.

44. The operation of Division is expressed by means of a sign, which is ÷ and sometimes :, and read *by*, or *divided by*; thus,

$$42 \div 7 = 6$$

denotes that the result of the division of 42 by 7 is 6: again, $(70 - 7) \div (4 + 5)$ is equivalent to $63 \div 9 = 7$; and the same kind of notation may easily be extended to quantities of much greater complexity.

MEASURES AND MULTIPLES.

45. In concluding the present chapter, it may not be improper to explain the meaning of certain terms hereafter made use of, and of frequent occurrence in the study of Mathematics, as well as to take notice of two Rules which seem naturally entitled to be considered here.

46. DEF. 1. A *Measure* of any number is one which will divide it without a remainder; as 5 is a measure of 15, because it is contained exactly 3 times in 15: but the element of number or 1, being a measure of every number, is never treated as a measure in whole numbers. It is said to *measure* the number, by the units contained in the quotient. All numbers whereof 2 is a measure are called *even* numbers, admitting of being divided into two *equal* parts, and all others are termed *odd* numbers.

47. DEF. 2. A *Common Measure* of two or more numbers is one, which will divide each of them without leaving a remainder; and the greatest of such measures is called the *Greatest Common Measure*, or *Greatest Common Divisor:* thus, 3 is *a* common measure of 18 and 30; whereas 6 is their *greatest* common measure, being the greatest number capable of dividing each of them without a remainder.

48. DEF. 3. An *Aliquot Part* of a number is any measure of it; in contradistinction to which a number which does not measure it exactly is sometimes called an *Aliquant Part*, by the old writers.

49. DEF. 4. A *Multiple* of any number is one which is divisible by it, or contains it a certain number

of times exactly; as 108 is a multiple of 12, because 12 is contained exactly 9 times in 108.

50. DEF. 5. A *Common Multiple* of two or more numbers is one which is divisible by each of them separately; and the *Least Common Multiple* is the least number that can be divided by each of them without a remainder: as 24 is *a* common multiple of 3 and 4, because divisible by both of them; whereas 12 is their *least* common multiple, because it is the least number that both 3 and 4 can divide without leaving remainders.

51. DEF. 6. A *Composite Number* is one which arises from the multiplication of *two* or *more* other numbers termed *Factors;* and it is thus distinguished from an *incomposite* or *prime* number, which cannot so originate: as 22 is a composite number, because it is equal to the product of the factors 2 and 11; but 11 is an incomposite or prime number, because the multiplication of no two or more factors will produce it, *unity*, which is merely the element of number, being excepted.

52. *If one number measure each of two others, it will also measure their sum, difference, and any multiple of each.*

Thus, 4 is a common measure of 20 and 12; and

$$\text{their sum} = 20 + 12 = 32 = 4 \times 8:$$
$$\text{their difference} = 20 - 12 = 8 = 4 \times 2:$$
$$\text{a multiple of } 20 = 20 \times 5 = 100 = 4 \times 25:$$
$$\text{a multiple of } 12 = 12 \times 7 = 84 = 4 \times 21:$$

each of which evidently comprises the measure 4 as a factor: and similarly of more numbers.

53. *To find the greatest common measure of two numbers.*

Let the numbers proposed be 63 and 168: then resolving each of them into its factors, we have

$$63 = 7 \times 9 \;\; = 7 \times 3 \times 3:$$
$$168 = 7 \times 24 = 7 \times 3 \times 8:$$

and the greatest common measure is evidently 7×3 or 21, because 3 and 8 have no common factor: and employing the principles of the last article, we obtain the same result in the following form:

63) 168 (2
126
42) 63 (1
42
21) 42 (2
42

where 21 the last *Divisor* is the greatest common measure: and we have hence the following rule.

Rule for finding the greatest common Measure.

Divide the greater of the proposed numbers by the less, and then the divisor by the remainder: repeat this operation till there is no remainder, and the last divisor will be the greatest common measure.

To ascertain the greatest common measure of three or more numbers, find the greatest common measure of any two of them: then that of this greatest common measure and another of them: and so on to the last.

Examples of the Greatest Common Measure.

54. Find the greatest common measures,

(1)	Of 9 and 24.	Answer, 3.
(2)	Of 126 and 144.	Answer, 18.
(3)	Of 3556 and 3444.	Answer, 28.
(4)	Of 5187 and 5850.	Answer, 39.
(5)	Of 6441 and 10283.	Answer, 113.
(6)	Of 13667 and 14186.	Answer, 173.
(7)	Of 43365 and 44688.	Answer, 147.
(8)	Of 11050 and 35581.	Answer, 221.
(9)	Of 109056 and 179712.	Answer, 1536.
(10)	Of 16, 24 and 140.	Answer, 4.
(11)	Of 945, 1560 and 22683.	Answer, 3.
(12)	Of 204, 1190, 1445 and 2006.	Answer, 17.

55. The following remarks will be of service to the student in making use of the last rule.

If the figure in the units' place be divisible by 2, the number is divisible by 2.

If the figures in the units' and tens' places be 4, the number is divisible by 4.

If the figures in the units', tens' and hundreds' places be 8, the number is divisible by 8.

If the sum of all the figures be divisible by 3 or 9, the number is divisible by 3 or 9.

If the figure in the units' place be 5 or 0, the number is divisible by 5.

If the sums of the alternate figures beginning at either end be equal, or one sum exceed the other by 11, or by any multiple of it, the number is divisible by 11.

56. *To find the least common multiple of two numbers.*

To find the least common multiple of 18 and 30, we observe that

$$18 = 6 \times 3 \text{ and } 30 = 6 \times 5,$$

so that the least number which contains them both exactly is evidently $6 \times 3 \times 5 = 90$, or the product of 18 and 30 divided by 6 their greatest common measure: and hence, the following rule.

Rule for finding the least common Multiple.

Divide the product of the two numbers by their greatest common measure, and the quotient will be their least common multiple.

If there be more than two numbers, proceed in the same way with the least common multiple of any two of them and the third: and so on, till they are all taken.

Examples of the Least Common Multiple.

57. Required the least common multiples,

(1) Of 12 and 27. Answer, 108.

(2) Of 289 and 323. Answer, 5491.

(3) Of 849 and 1132. Answer, 3396.

(4) Of 3, 4 and 16. Answer, 48.

(5) Of 24, 39 and 376. Answer, 14664.

(6) Of 12, 15, 35 and 56. Answer, 840.

58. In the application of the rule last given, the process will be made easier by multiplying either of the numbers proposed by the quotient arising from the division of the other by their greatest common measure: as the result will evidently be the same by each method.

General proofs of all that has been said here, may be found in the Author's *Elements of Algebra.*

CHAPTER II.

APPLICATION OF ARITHMETIC TO NUMERICAL MAGNITUDES OF VARIOUS DENOMINATIONS, NOT CONNECTED BY THE BASE OF THE COMMON SYSTEM OF NOTATION.

59. In the preceding Chapter we have considered only such *abstract* numbers as are formed by figures whose local values are always regulated by the same fixed number *ten*, which is called the *Base* of the Common System of Notation: but the rules hitherto given, are easily extended to *concrete* magnitudes wherein the local values of the different figures are connected by more numbers than one; as for instance, to *Pounds*, *Shillings*, *Pence* and *Farthings*, where *four* farthings are equivalent to *one* penny, which is the next higher denomination; *twelve* pence to *one* shilling, which is the next denomination in order; and *twenty* shillings to *one* pound, which is the highest denomination here mentioned; the *different* numbers 4, 12 and 20 connecting the different denominations, in precisely the same manner as the *fixed* number 10, was supposed to connect the successive denominations of Integers.

The processes employed in cases of this nature are *Reduction*, and the fundamental operations then called *Compound Addition*, *Compound Subtraction*, *Compound Multiplication* and *Compound Division*; each of which will be exemplified in order, and the *Tables* by means of which they are conducted, will be found at the beginning of the work.

REDUCTION.

60. Def. *Reduction* is the conversion or changing of numerical quantities, from one or more denominations to one or more others, such that the real or absolute values shall remain unaltered: and its operations will evidently depend upon the principles already explained.

Ex. To reduce £25. 13*s.* 6¾*d.* into farthings, and conversely.

The correctness of the following operations will be manifest from the explanations annexed to their several steps, which are omitted as unnecessary in *practice:*

Direct Operation.

£.	*s.*	*d.*
25	13	6¾

```
  £.   s.  d.
  25 . 13 . 6¾
   2 0 s. = 1 pound:
  ——          £.   s.   d.
 5 1 3 s. = 25 . 13 . 0.
   1 2 d. = 1 shilling:
 6 1 6 2 d. = 25 . 13 . 6.
     4 f. = 1 penny:
2 4 6 5 1 f. = 25 . 13 . 6¾.
```

Converse Operation.

```
                   far.
1d. = 4f. ) 2 4 6 5 1
1s. = 12d. ) 6 1 6 2¾
 £1. = 20s. ) 5 1 3 . 6
           £ 25 . 13 . 6¾.
```

Here the denominations are separated by a point as (.); and this is necessary to distinguish them from *ordinary* numbers, which do not require it because their local values are all fixed and certain: and it is moreover evident that each of these operations may be regarded as a *proof* of the other.

61. The former process is sometimes called a *descending* and the latter an *ascending* reduction, and they lead respectively to the following rules.

RULE I. To reduce quantities from *higher* to *lower* denominations, *multiply* the highest denomination by the number which connects it with the next inferior, and to the product add the number of the inferior denomination in the quantity proposed; and repeat this for each succeeding denomination till the required one is obtained.

RULE II. To reduce quantities from *lower* to *higher* denominations, *divide* them by the numbers which connect the different denominations in order, and annex the remainders at each step, so as to retain the denominations of the dividends from which they respectively arise.

Ex. How many half-crowns are equivalent to £253. 9*s.* 10*d.*?

Here both the rules are requisite, and we have the following operation:

$$\begin{array}{rl}
& \text{£.} \quad s. \quad d. \\
& 253 \,.\, 9 \,.\, 10 \\
& \quad 20 \\
\hline
& 5069\, s. \\
& \quad 12 \\
\hline
1 \text{ half-crown} = 30d.) & 60838\, d. \\
\hline
\text{half-crowns} & 2027 \,.\, 28d:
\end{array}$$

that is, the proposed sum is equivalent to 2027 half-crowns with 28*d.* or 2*s.* 4*d.* remaining; and this result is verified by reversing the process: thus,

$$\begin{array}{rl}
& \textit{half c.} \quad d. \\
& 2027 \,.\, 28. \\
& \quad 30 \\
\hline
12\,) & 60838\, d. \\
\hline
20\,) & 5069 \,.\, 10d. \\
\hline
& \text{£}253 \,.\, 9 \,.\, 10.
\end{array}$$

Examples for Practice.

(1) Reduce £71. 13*s.* 6½*d.* into farthings; and verify the result.

Answer: 68810 farthings.

(2) Find the number of farthings in 95 guineas 17*s.* 9¾*d*: and conversely.

Answer: 96615 farthings.

(3) Reduce £295. 18*s.* 3¾*d.* to farthings; and prove it.

Answer: 284079 farthings.

(4) Find the number of pounds, &c., in 415739 farthings; and prove it.

Answer: £433. 1*s.* 2¾*d.*

(5) Reduce 14cwt. 3qrs. 24lbs. into ounces; and prove it.

Answer: 26816 ounces.

(6) Find the number of ounces in 11cwt. 2qrs. 17lbs. 15oz.; and prove it.

Answer: 20895 ounces.

(7) What number of cwts., &c., are contained in 65437 drams? and verify the result.

Answer: 2cwt. 1qr. 3lbs. 9oz. 13drs.

(8) Reduce 3tons. 14cwt. 3qrs. 25lbs. 11oz. 9drs. into drams; and prove the result.

Answer: 2149817 drams.

(9) Find the number of poles contained in 15mi. 5fur. 31po.; and verify the result.

Answer: 5031 poles.

(10) In 1081080 inches, how many miles, &c.,? and prove it.

Answer: 17 miles, 110 yards.

(11) Reduce 304935 feet to miles, &c.,; and the converse.

Answer: 57mi. 6fur. 5yds.

(12) What number of inches are equivalent to 512yds. 2ft. 9in.? and prove the result.

Answer: 18465 inches.

(13) Reduce 54yds. 8ft. 104in., superficial measure, into inches.

Answer: 71240 inches.

(14) What number of superficial yards, &c., are equivalent to 40253798 superficial inches?

Answer: 31060yds. 38in.

(15) Find the number of cubic yards, &c., in 141721 cubic inches; and prove it.

Answer: 3yds. 1ft. 25in.

(16) In 5279 pints, how many gallons, &c.,? and prove the result.

Answer: 659gals. 3qts. 1pt.

(17) Required the number of weeks, &c., in 72015 hours; and verify the result.

Answer: 428wks. 4days. 15hrs.

(18) In 2706359 seconds, how many weeks, &c.,? and prove it.

Answer: 4wks. 3days. 7hrs. 45min. 59sec.

(19) How many degrees, &c., are of equal value with 206265 seconds? and prove the converse.

Answer: $57^{\circ} . 17' . 45''$.

(20) In 340 pistoles at 17*s.* 6*d.* each, how many pounds sterling?

Answer: £297 . 10*s.*

(21) How many moidores of 27*s.* each, are equal to 198 guineas?

Answer: 154 moidores.

(22) In 12lbs. 10oz. 15dwts. 14grs. of silver, how many grains? and prove the result.

Answer: 74294 grains.

(23) Find how many grains there are in 18lbs. 2oz. 4drs. 2scr. 12grs; and give a proof.

Answer: 104932 grains.

(24) In 20 yds. 3qrs. 1nl., find the number of nails; and prove it.

Answer: 333 nails.

(25) What number of acres, &c., are equal in extent to 82973 square poles?

Answer: 518ac. 2ro. 13po.

(26) How many pints are equivalent to 987bar. 25gals. 3qts. 1pt. of ale? and prove it.

Answer: 284463 pints.

(27) Reduce 21tuns. 3hhds. 54gals. 2qts. of wine to pints; and the contrary.

Answer: 44284 pints.

(28) Required the number of quarts in 356qrs. 7bu. 2pks. 1gal. of corn; and prove it.

Answer: 91380 quarts.

(29) In £453. 16*s.* 8*d.*, how many pieces of coin valued at 3*s.* 4*d.* each?

Answer: 2723 pieces.

(30) Find how often a rod of 2ft. 10in. in length must be applied to measure 10 miles, 140 yards.

Answer: 18783 times, and 18in. over.

(31) What number of weights of 14oz. 13drs. each, are equivalent to 25cwt. 2qrs. 14lbs?

Answer: 3100 weights, and 1oz. 4drs. over.

(32) How many revolutions will the wheel of a carriage, 4ft. 7in. in circumference, make in 2mi. 4fur.?

Answer: 2880 revolutions.

(33) If 5oz. of silk can be spun into a thread 2fur. 20po. long; what weight of silk would supply a thread sufficient to reach to the moon, if the distance be 240000 miles?

Answer: 107tons. 2cwt. 3qrs. 12lbs.

(34) A year being equivalent to 365 days 6 hours, it is required to find the number of years, &c., in 295402374 seconds.

Answer: 9yrs. 131days. 18hrs. 12min. 54sec.

62. Keeping in mind what was said in the first article of this chapter, we need no additional enquiry to inform us that the fundamental operations on *Compound Quantities* must be performed as in *Integers*, with this difference, that instead of carrying and borrowing *tens*, we must do the same with the *different numbers* which connect their parts together: and we shall therefore merely enunciate the rule for each, at the beginning of the portion of the work appropriated to it.

I. COMPOUND ADDITION.

Rule. Arrange the quantities under one another according to their denominations: add together those of the lowest, and having found the number of the next denomination to which the sum is equivalent, put down the remainder, if any, and add this number to those of the next denomination; and repeat the process till all the quantities are disposed of.

Ex. Find the sum of 142cwt. 1qr. 21lbs., 78cwt. 0qr. 14lbs., 21cwt. 2qrs. 19lbs., and 176cwt. 1qr. 15lbs.

The form of the operation is as underneath:

Common Operation.

cwt.	qrs.	lbs.	
142	1	21	
78	0	14	
21	2	19	
176	1	15	
cwt. 418	2	13	the sum:

Reductions.

lbs. lbs. qrs.
28) 69 (2
56
13 lbs:

qrs. qrs. cwt.
4) 6 (1
4
2 qrs:

and the method of proof is similar to that of Simple Addition in article (23).

Examples for Practice.

(1) Add together £73. $2s.$ $9\frac{1}{2}d.$; £25. $8s.$ $4\frac{3}{4}d.$; £68. $3s.$ $11\frac{1}{4}d.$; £76. $17s.$ $7d.$ and £5. $14s.$ $5\frac{3}{4}d.$: and prove the result.

Answer: £249. $7s.$ $2\frac{1}{4}d.$

(2) Find the sum of 32 cwt. 2 qrs. 15 lbs. 12 oz.; 47 cwt. 25 lbs. 7 oz.; 5 cwt. 3 qrs. 17 lbs. 10 oz.; 23 cwt. 1 qr. 19 lbs. 15 oz.; 1 cwt. 2 qrs. 10 lbs. 8 oz., and 9 cwt. 3 qrs. 14 oz.: and prove it.

Answer: 120 cwt. 2 qrs. 6 lbs. 2 oz.

(3) Required the sum of 11 yds. 2 ft. 9 in.; 46 yds. 1 ft. 8 in.; 15 yds. 1 ft. 10 in.; 38 yds. 2 ft. 9 in.; 55 yds. 11 in., and 27 yds. 2 ft. 7 in.: and prove it.

Answer: 196 yds. 6 in.

(4) Collect into one quantity 49 gals. 3 qts. 1 pt.; 34 gals. 1 qt.; 25 gals. 1 pt.; 51 gals. 3 qts. 1 pt.; 30 gal. 1 qt., and 53 gals. 2 qts. 1 pt.: and prove it.

Answer: 245 gallons.

(5) Determine the aggregate of 10 wks. 5 days. 14 hrs. 31 min.; 18 wks. 4 days. 12 hrs. 38 min.; 25 wks. 10 hrs. 14 min.; 75 wks. 6 days. 23 hrs. 59 min.; 53 wks. 4 days. 19 hrs. 23 min., and 40 wks. 17 hrs. 25 min.: and prove it.

Answer: 224 wks. 2 days. 2 hrs. 10 min.

(6) Add together 64 lbs. 11 oz. 16 dwts. 14 grs.; 21 lbs. 10 oz. 12 dwts. 13 grs.; 2 lbs. 1 dwt. 16 grs.; 12 lbs. 10 oz.

18grs.; 24lbs. 11oz. 12dwts., and 14lbs. 1oz. 1gr.: and prove the result.

Answer: 140lbs. 9oz. 3dwts. 14grs.

(7) Find the sum of 11oz. 4drs. 2scrs. 11grs.; 10oz. 3drs. 4grs.; 11oz. 1scr. 14grs.; 10oz. 1scr. 16grs.; 2drs. 2scrs. 18grs., and 14oz. 5drs. 1scr.: and prove it.

Answer: 58oz. 1dr. 1scr. 3grs.

(8) Express in one sum, 21yds. 2qrs. 3nls.; 18yds. 2qrs. 2nls.; 21yds. 1qr. 2nls.; 16yds. 3qrs. 2nls.; 12yds. 1qr. 2nls., and 14yds. 2qrs. 3nls.

Answer: 105yds. 2qrs. 2nls.

(9) Find the sum of 32lea. 2mi. 1fur. 21po.; 16lea. 1mi. 3fur. 26po.; 18lea. 2mi. 6fur. 21po.; 13lea. 1mi. 2fur. 12po., and 26lea. 1mi. 4fur. 9po.

Answer: 108lea. 2fur. 9po.

(10) Find the sum of 21ac. 1ro. 34po.; 16ac. 2ro. 27po.; 214ac. 1ro. 2po.; 32ac. 1ro. 28po. and 301ac. 14po.

Answer: 585ac. 3ro. 25po.

II. COMPOUND SUBTRACTION.

Rule. Having properly arranged the quantities under one another, begin at the right hand and take each number in the lower line from the corresponding one in the upper, borrowing instead of 10, when necessary, the numbers which connect the successive denominations together; and the several quantities thus obtained will be the difference required.

Ex. Let it be required to subtract 35yds. 2ft. 8in., from 48yds. 1ft. 4in.

Common Operation.

yds.	ft.	in.	
48	1	4	
35	2	8	
yds. 12	1	8	the diff.

Reductions.

$$\begin{aligned} \text{in.}\quad & \quad \text{in.} \quad\ \ \text{in.} \\ 16 &= 4 + 12 \text{ borrowed,} \\ 8 & \\ \hline 8 & \text{ in.} \end{aligned}$$

$$\begin{aligned} \text{ft.}\quad & \quad \text{ft.} \quad\ \ \text{ft.} \\ 4 &= 1 + 3 \text{ borrowed,} \\ 3 &= 2 + 1 \text{ carried,} \\ \hline 1 & \text{ ft.} \end{aligned}$$

and the proof used for integers is applicable here.

Examples for Practice.

(1) Find the difference of £325. 19*s*. 4¾*d*. and £253. 18*s*. 6½: and prove it.

Answer: £72. 0*s*. 10¼*d*.

(2) Required the excess of 59tons. 13cwt. 2qrs. 23lbs. 11oz. 10drs. above 27tons. 17cwt. 1qr. 25lbs. 2oz. 14drs.; and verify the result.

Answer: 31tons. 16cwt. 26lbs. 8oz. 12drs.

(3) Subtract 82lea. 2mi. 5fur. 38po. from 281lea. 1mi. 7fur. 26po.; and verify it.

Answer: 198lea. 2mi. 1fur. 28po.

(4) Find the difference of 140gals. 3qts. 1pt. and 240gals.; and prove it.

Answer: 99gals. 1pt.

(5) From 24days. 14hrs. 46min. 31sec. take 4days. 21hrs. 18min. 52sec.; and verify it.

Answer: 19days. 17hrs. 27min. 39sec.

(6) Take 14lbs. 11oz. 12dwts. 19grs. from 81lbs. 10oz. 9dwts. 18grs.

Answer: 66lbs. 10oz. 16dwts. 23grs.

(7) Required the difference of 28lbs. 7oz. 1dr. 2scr. 4grs. and 12lbs. 8oz. 2drs. 1scr. 12grs.

Answer: 15lbs. 10oz. 7drs. 12grs.

(8) What is the difference between 38ac. 31po. and 21ac. 3ro. 34po.?

Answer: 16ac. 37po.

(9) Subtract 1tun. 3hhds. 32gals. 4pts of wine from 3tuns. 2hhds.

Answer: 1tun. 2hhds. 30gals. 4pts.

(10) Required the difference of 162qrs. 1bush. 1pk. and 127qrs. 4bush. 3pks. 1gal.

Answer: 34qrs. 4bush. 1pk. 1gal.

III. COMPOUND MULTIPLICATION.

RULE. Place the multiplier under the lowest denomination of the multiplicand, and find the number of the next denomination contained in the first product: put down the remainder, if any, and carry the quotient to

the second product, and repeat the process till all the denominations are multiplied; and thus the required product will be determined.

Ex. Multiply 35gals. 3qts. 1pt. by 7.

Common Operation.	*Reductions.*
gals. qts. pt.	pts. pts. qts. qts.
35 . 3 . 1	2) 7 4) 24
7	
gals. 251 . 0 . 1 the prod:	qts. 3 . 1 pt: gals. 6 . 0 qt:

and this may be proved by reducing 35gals. 3qts. 1pt. to pints, multiplying the result by 7, and then reducing the product to gallons, &c.

63. When the multiplier exceeds 12, this process would be laborious, and it is usual to extend what was said in (35) to such cases.

Ex. To find the product of 3days. 18hrs. 45min. by 47, we may use either of the following operations:

days. hrs. min.	days. hrs. min.
3 . 18 . 45	3 . 18 . 45
$9 \times 5 + 2 = 47$	$8 \times 6 - 1 = 47$
34 . 0 . 45	30 . 6 . 0
5	6
170 . 3 . 45	181 . 12 . 0
7 . 13 . 30	3 . 18 . 45
da. 177 . 17 . 15 the prod.	da. 177 . 17 . 15 the prod.

If however this method require many factors to make up the multiplier, it is best to reduce the multiplicand to the lowest denomination contained in it, to multiply this result by the multiplier, and then to reduce the product back again.

Examples for Practice.

(1) Multiply £358. 4*s*. 7¾*d*. by 5 and 9.

Answers: £1791. 3*s*. 2¾*d*., and £3224. 1*s*. 9¾*d*.

(2) Required the products of 49cwt. 3qrs. 15lbs. by 7 and 11.

Answers: 349cwt. 21lbs., and 548cwt. 2qrs. 25lbs.

(3) Find the products of 154yds. 2ft. 10in. by 6 and 10.

Answers: 929yds. 2ft., and 1549yds. 1ft. 4in.

(4) Multiply 58gals. 3qts. 1pt. by 8 and 12.

Answers: 471gals., and 706gals. 2qts.

(5) Multiply 42wks. 5days. 23hrs. 42min. by 3 and 4.

Answers: 128wks. 3days. 23hrs. 6min., and 171wks. 2days. 22hrs. 48min.

(6) Multiply £125. 15*s*. 9¼*d*. by 28 and 45.

Answers: £3522. 1*s*. 7*d*., and £5660. 9*s*. 8¼*d*.

(7) Multiply £53. 18*s*. 7¾*d*. by 51 and 83.

Answers: £2750. 10*s*. 11¼*d*., and £4476. 7*s*. 7¼*d*.

(8) Multiply 17cwt. 2qrs. 19lbs. 5oz. by 36 and 73.

Answers: 636cwt. 23lbs. 4oz., and 1290cwt. 9lbs. 13oz.

(9) Multiply 13lea. 2mi. 6fur. 25po. by 42 and 97.

Answers: 585lea. 1mi. 6fur. 10po., and 1352lea. 1mi. 2fur. 25po.

(10) Multiply 15qrs. 6bush. 3pks. 1gal. by 54 and 111.

Answers: 856qrs. 3bush. 1pk., and 1760qrs. 3bush. 1gal.

(11) Multiply 43days. 18hrs. 45min. by 77 and 147.

Answers: 3371days. 3hrs. 45min., and 6435days. 20hrs. 15min.

(12) Multiply 57°. 7′. 45″ by 4 and 6.

Answers: 228°. 31′, and 342°. 46′. 30″.

(13) What is the price of 72 reams of paper, at 13*s*. 8*d*. a ream?

Answer: £49. 4*s*.

(14) Find the price of 120 ounces of silver, at 5*s*. 3¾*d*. an ounce.

Answer: £31. 17*s*. 6*d*.

(15) Find the number of yards in 40 pieces of cloth, each containing 42yds. 2qrs. 2nls.

Answer: 1705 yards.

(16) Required the price of 279cwt. at £3. 7*s*. 10½*d*. a cwt.

Answer: £946. 17*s*. 1½*d*.

(17) If I spend £2. 7*s*. 1½*d*. a day, how much is that in a year of 365 days?

Answer: £860. 0*s*. 7½*d*.

(18) What sum will purchase an estate of 2120 acres, when the price of each acre is £32. 5*s*. 6*d*.?

Answer: £68423.

(19) If each of 114 persons receive £1. 18*s*. 6½*d*., what is received by them all?

Answer: £219. 13*s*. 9*d*.

(20) How many pounds of silver are there in a half-dozen of dishes, each weighing 51oz. 10dwts., and a dozen of plates each weighing 15oz. 15dwts. 22grs.?

Answer: 41lbs. 6oz. 11dwts.

(21) If a wheel of 5yds. 1ft. 6in. in circumference make 64640 revolutions, what space will it pass over?

Answer: 202 miles.

IV. COMPOUND DIVISION.

Rule. Having placed the divisor and dividend as in integers, find how often the divisor is contained in the highest denomination of the dividend, put down the quotient and reduce the remainder, if any, to the next inferior denomination, adding to it the number of that denomination in the dividend, and repeat the division: proceed thus through all the denominations, and the entire quotient will be obtained.

Ex. Divide 41wks. 6days. 19hrs. by 11.

Common Operation.

$$\begin{array}{r|l} 11 & \overset{\text{wks.}}{41} \;.\; \overset{\text{days.}}{6} \;.\; \overset{\text{hrs.}}{19} \\ \hline \text{wks.} & 3 \;.\; 5 \;.\; 17 \text{ the quot:} \end{array}$$

Reductions.

$$\begin{array}{r|l} 11 & \overset{\text{wks.}}{41} \\ \hline \text{wks.} & 3 \;.\; 8 \text{ weeks over:} \end{array}$$

$$\begin{array}{r|l} 11 & \overset{\text{days.}}{62} = 8 \times \overset{\text{days.}}{7} + 6, \\ \hline \text{days} & 5 \;.\; 7 \text{ days over:} \end{array}$$

$$\begin{array}{r|l} 11 & \overset{\text{hrs.}}{187} = 7 \times \overset{\text{hrs.}}{24} + 19, \\ \hline \text{hrs.} & 17: \end{array}$$

and the operation may be proved by that of multiplication.

64. When the divisor is greater than 12, the process may be conducted as in (42), if it be a composite number, and by long division, if it be incomposite.

Ex. To divide £1478. 13*s*. 8$\frac{3}{4}$*d*. into 77 equal portions, we may use either of the subjoined methods:

$$77\left\{\begin{array}{rrrrl} & \pounds. & s. & d. & \\ 7) & 1478\,. & 13\,. & 8\frac{3}{4} & \\ \hline 11) & 211\,. & 4\,. & 9\frac{3}{4}\,. & 2 \\ \hline \pounds & 19\,. & 4\,. & 0\frac{3}{4}. & 44\text{f.over.} \end{array}\right.$$

$$\begin{array}{rl} & \pounds. \quad s. \quad d. \quad \pounds. \quad s. \quad d. \\ 77) & 1478\,.\,13\,.\,8\frac{3}{4}.\,(19\,.\,4\,.\,0\frac{3}{4} \\ & 77 \\ \hline & 708 \\ & 693 \\ \hline & 15 \\ & 20 \\ \hline & 313 \\ & 308 \\ \hline & 5 \\ & 12 \\ \hline & 68 \\ & 4 \\ \hline & 275 \\ & 231 \\ \hline & 44 \text{ f. over.} \\ \hline \end{array}$$

The division may also be effected by reductions analogous to those alluded to in Multiplication.

Examples for Practice.

(1) Divide £189. 8*s*. 4*d*. by 5 and 8.

Answers: £37. 17*s*. 8*d*., and £23. 13*s*. 6$\frac{1}{2}$*d*.

(2) Required the quotients of 182cwt. 3qrs. 7lbs. by 7 and 9.

Answers: 26cwt. 13lbs., and 20cwt. 1qr. 7lbs.

(3) Divide 1658yds. 1ft. by 6 and 10.

Answers: 276yds. 1ft. 2in., and 165yds. 2ft. 6in.

(4) Find the quotients of 288 ac. 2 ro. 32 po. by 8 and 11.

Answers: 29 ac. 3 ro. 14 po., and 21 ac. 2 ro. 32 po.

(5) Divide 13 wks. 5 days. 19 hrs. 30 min. by 3 and 4.

Answers: 4 wks. 4 days. 6 hrs. 30 min., and 3 wks. 3 days. 4 hrs. 52 min. 30 sec.

(6) Divide £1738. 12*s*. 7½*d*. by 18, and £1279. 13*s*. 8¾*d*. by 23.

Answers: £96. 11*s*. 9¾*d*., and £55. 12*s*. 9¼*d*.

(7) Divide 425 tons. 15 cwt. 2 qrs. 12 lbs. by 27, and 2374 cwt. 1 qr. 12 lbs. 12 oz. by 38.

Answers: 15 tons. 15 cwt. 1 qr. 16 lbs., and 62 cwt. 1 qr. 26 lbs. 2 oz.

(8) Find the quotient of 1361 mi. 4 fur. 28 po. by 28, and of 3179 lea. 1 mi. 5 fur. 16 po. by 46.

Answers: 48 mi. 5 fur. 1 po., and 69 lea. 2 fur. 36 po.

(9) Divide 739 qrs. 4 bush. 2 pks. 1 gal. into 11 equal portions.

Answer: 67 qrs. 1 bush. 3 pks. 1 gal.

(10) What is the twelfth part of 22 wks. 4 days. 20 hrs. 43 min. 24 sec.?

Answer: 1 wk. 6 days. 5 hrs. 43 min. 37 sec.

(11) If 41 cwt. cost £52. 10*s*. 7½*d*., what is the price of a cwt.?

Answer: £1. 5*s*. 7½*d*.

(12) What will be the price of 1 lb., when 1 cwt. costs £137. 18*s*.?

Answer: £1. 4*s*. 7½*d*.

(13) If a soldier's pay for a year of 365 days be £9. 2*s*. 6*d*., how much is that for a day?

Answer: 6*d*.

(14) If a person's yearly income be £65. 12*s*. 6*d*., and he lay by £20. a year, how much does he spend each day?

Answer: 2*s*. 6*d*.

(15) If 145 sheep cost £169. 5*s*. 9*d*., what is the price of a score at the same rate?

Answer: £23. 7*s*.

(16) A wheel makes 514 revolutions in passing over 1 mi. 467 yds. 1 ft., what is its circumference?

Answer: 4 yds. 1 ft.

(17) If a person complete a journey of 422 mi. 3 fur. 38 po. in 37 days, what distance does he travel each day?

Answer: 11 mi. 3 fur. 14 po.

(18) If 8 packages of cloth, each consisting of 4 parcels, each parcel of 10 pieces, and each piece of 26 yards, cost £6656., what is the price of a yard?

Answer: 16 shillings.

(19) If the clothing of 754 soldiers come to £3178. 11*s.* $7\frac{1}{2}$*d.*, how much is that for each man?

Answer: £4. 4*s.* $3\frac{3}{4}$*d.*

(20) A vintner bought 138 gals. at 10*s.* a gallon, of which he retained 18 gals. for his own use: at what rate per gallon must he sell the remainder, that he may have his own for nothing?

Answer: 11*s.* 6*d.*

65. The multipliers and divisors in the last two rules have always been regarded as abstract numbers: and though it may not be generally possible to determine the product of two concrete quantities, the quotient of one concrete magnitude by another of the same kind will be an abstract number.

Ex. Find how often £37. 12*s.* $8\frac{1}{2}$*d.*, is contained in £263. 8*s.* $11\frac{1}{2}$*d.*

Here the dividend = 252910 farthings:

and the divisor = 36130 farthings:

whence the quotient is found to be 7 by common division: or £37. 12*s.* $8\frac{1}{2}$*d.* being *repeated* 7 times, amounts to £263. 8*s.* $11\frac{1}{2}$*d.*

Hence one concrete magnitude may be a measure or a multiple of another of the same kind.

CHAPTER III.

THE RULE OF THREE,

SOMETIMES CALLED THE GOLDEN RULE.

66. DEF. THE object of the *Rule of Three* is, by means of *three* quantities given, to determine a *fourth*, which shall be the same multiple, part or parts of one of them, that one of the remaining quantities is of the other; and it therefore follows, that the operation by which this may be accomplished, will depend upon those of Multiplication and Division already considered.

Ex. 1. If 1 lb. of any commodity cost 3*s*. $4\frac{1}{2}$*d*., it is required to find the price of 12 lbs.

Here it is evident that the required price will be the same multiple of 3*s*. $4\frac{1}{2}$*d*., that 12 lb. is of 1 lb., which may therefore be found by Compound Multiplication; thus,

$$\begin{array}{rrl} & s.\quad d. & \\ & 3\ .\ 4\tfrac{1}{2} & = \text{price of 1 lb.} \\ & 12 & \\ \hline \pounds\,2\ .\ 0\ .\ 6 & & = \text{price of 12 lbs.:} \end{array}$$

and the same result may be obtained by means of a *statement* and *operation* in the following form:

$$\begin{array}{r} \text{lb.}\quad \text{lbs.}\quad\ s.\quad d. \\ 1\ :\ 12\ ::\ 3\ .\ 4\tfrac{1}{2} \\ 12 \\ \hline 40 \\ 4 \\ \hline 162 \\ 12 \\ \hline 4\,)\,1944\,f. \\ \hline 12\,)\,486\,d. \\ \hline 20\,)\,40\ 6d. \\ \hline \pounds\,2\ .\ 0\ .\ 6.\ \text{as before.} \end{array}$$

Ex. 2. If 11 bushels of wheat cost £4. 2*s*. $11\frac{1}{3}$*d*., what sum must be paid for 45 bushels?

In this instance, we have, by Division,

$$\begin{array}{r|l} & \pounds.\ \ s.\ \ d. \\ 11 & 4 \ .\ 2 \ .\ 11\frac{1}{3} \\ \hline & \pounds\ 0 \ .\ 7 \ .\ 6\frac{1}{3} = \text{price of 1 bushel:} \end{array}$$

and then by Multiplication, the required price is obtained thus:

$$\begin{array}{rrrl} \pounds. & s. & d. & \\ 0 \ . & 7 \ . & 6\frac{1}{3} & \\ & & 9 & \times 5 = 45 \\ \hline 3 \ . & 7 \ . & 10\frac{1}{3} & \\ & & 5 & \\ \hline \pounds\ 16 \ . & 19 \ . & 4\frac{1}{3} & = \text{price of 45 bushels:} \end{array}$$

and we shall arrive at the same conclusion by working the question in a form similar to that of the last example: as,

$$\begin{array}{r} \text{bush.} \quad \text{bush.} \quad \pounds. \ \ s. \quad d. \\ 11 : 45 :: 4 \ .\ 2 \ .\ 11\frac{1}{3} \\ 20 \\ \hline 82 \\ 12 \\ \hline 995 \\ 4 \\ \hline 3982 \\ 45 \\ \hline 19910 \\ 15928 \\ \hline 11\,)\,179190 \\ \hline 4\,)\,16290 \\ \hline 12\,)\,4072\frac{1}{3} \\ \hline 2{,}0\,)\,33{,}9 \ .\ 4 \\ \hline \pounds\ 16 \ .\ 19 \ .\ 4\frac{1}{3}: \end{array}$$

and it is easily shown that this sum consists of the same multiple and part of £4. 2*s.* 11½*d.*, that 45 does of 11.

67. Proper attention to the principles employed in these two Examples, will enable us to embody their substance in the following general rule.

RULE OF THREE.

For the Statement. Of the three quantities proposed, put down as the last, that which is of the *same kind,* or under the *same circumstances* as the one required; and the *greater* or *less* of the two others in the second place, according as the required one ought, from the nature of the case, to be *greater* or *less* than the last; and the remaining one in the first place.

For the Operation. Reduce, if necessary, the first and second terms to the *same* denomination, and the third to the *lowest* denomination contained in it: multiply together the second and third terms thus reduced, and the quotient arising from the division of the product by the first, will be the quantity required, expressed in the *denomination* to which the *last* term was reduced: which may be had in other terms by the proper divisions or multiplications.

It is sometimes necessary to consider what *preparation* may be required before the rule is applied: and it is evident that when the statement is made, the first, and the second *or* third terms, may be divided by any factor common to them both, without affecting the result, inasmuch as no alteration is produced from Multiplication and Division by the same number.

Examples for Practice.

(1) Required the price of 450lbs., at 4*s.* 8½*d.* a lb.

Answer: £105. 18*s.* 9*d.*

(2) Find the amount of a servant's wages for 215 days, at 2*s.* 4¼*d.* a day.

Answer: £25. 6*s.* 1¾*d.*

(3) If 25cwt. 2qrs. cost £7. 6*s.* 7½*d.*, how much is that for 1 cwt.?

Answer: 5*s.* 9*d.*

(4) Required the price of 4cwt. 1qr. 4lbs. 8oz., when 1lb. costs 7*s.* 10½*d.*

Answer: £189. 3*s.* 11¼*d.*

(5) If 6 yds. 3 qrs. cost 5*s.* 3*d.*, how much will 73 yds. 2 qrs. cost, at the same rate?

Answer: £2. 17*s.* 2*d.*

(6) If an artificer earn £19. 1*s.* in 20 days; in what time will he earn £23. 16*s.* 3*d.*?

Answer: 25 days.

If he performs a piece of work in 7 days.

(7) If a person ~~walk~~ 216 miles in 7 days, of 16 hours each; in how many days of 12 hours each can he do the same?

Answer: 9 days 4 hours.

(8) If 17 ells. 3 qrs., each ell containing 5 qrs., be bought for £6. 17*s.* 6*d.*: how much must be paid for 18 yds.?

Answer: £5. 12*s.* 6*d.*

(9) If 12 quarts of wine cost £2. 5*s.*, it is required to find the price of 5 pipes.

Answer: £472. 10*s.*

(10) How much wheat can be purchased for £55. 0*s.* 3*d.*, at the rate of 6*s.* 9½*d.* a bushel?

Answer: 20 qrs. 2 bush.

(11) If a farm of 375 acres, be let for £401. 11*s.* 3*d.* a year, what is that for each acre?

Answer: £1. 1*s.* 5*d.*

(12) If lodgings be let at 13*s.* 6*d.* a week, what will the demand amount to for 273 days?

Answer: £26. 6*s.* 6*d.*

(13) Required the price of 36 cwt. 1 qr., when 2 cwt. 2 qrs. 10 lbs. cost £4. 7*s.* 9½*d.*?

Answer: £61. 9*s.* 1*d.*

(14) If a servant's wages be £30. 0*s.* 8¾*d.* a year, what will be his demand for a service of 338 days?

Answer: £27. 16*s.* 3½*d.*

(15) If a person can walk 3 mi. 6 fur. 25 po. in an hour, in what time will he complete a journey of 99 mi. 4 fur. 10 po.?

Answer: 26 hours.

(16) What is the cost of 19 bar. 24 gals. 3 qts. 1 pt. of beer, at 3½*d.* a quart?

Answer: £41. 7*s.* 0¼*d.*

(17) If the carriage of 3 cwt. 2 qrs. 14 lbs. for $51\frac{1}{2}$ miles come to 18*s*. $5\frac{1}{4}$*d*.; what will be the charge for carrying 10 tons. 3 cwt. the same distance?

Answer: £51. 12*s*. 6*d*.

(18) At the rate of 11*s*. $7\frac{1}{2}$*d*. in the pound, what is the sum paid by a bankrupt for a debt of £2735. 10*s*.?

Answer: £1590. 0*s*. $2\frac{1}{4}$*d*.

(19) If a labourer earn 2*s*. a day when wheat is at 8*s*. a bushel, what ought he to earn when wheat is at 6*s*. a bushel?

Answer: 1*s*. 6*d*.

(20) If a tradesman gain 1*s*. $4\frac{1}{2}$*d*. on an article which he sells for 5*s*. 6*d*., what does he gain on every £100.?

Answer: £25.

(21) If 15 workmen can do a piece of work in 25 days, in what time can 25 men do the same?

Answer: 15 days.

(22) How much in length, that is 3 ft. 9 in. broad, will be equivalent to 37 ft. 9 in. in length, which is 7 ft. 6 in. broad?

Answer: 75 ft. 6 in.

(23) If 69 yds. of carpet 3 qrs. wide, cover a room 8 yds. 2 qrs. 2 nls. long; find the width of the room.

Answer: 6 yards.

(24) What would be the purchase-money of an estate producing a rental of £3223., at the rate of £2. 15*s*. per cent?

Answer: £117200.

(25) What may a person, having an income of £1000. a year, spend daily, so as to lay by £434. 5*s*. yearly?

Answer: £1. 11*s*.

(26) If I lend a friend £250. for 6 months, how long ought he to lend me £187. 10*s*. to requite the kindness?

Answer: 8 months.

(27) If the rate levied upon a rental of £763. 15*s*. amount to £133. 13*s*. $1\frac{1}{2}$*d*., how much is that in the pound?

Answer: 3*s*. 6*d*.

(28) A person buys 136 yds. of cloth for £150., and retails it at £1. 18*s*. a yard; what does he gain by the transaction?

Answer: £108. 8*s*.

(29) A person's daily income is £1. 15*s*. and his quarterly expenditure £135. 10*s*.: how much will he have saved at the end of 9 years?

Answer: £870. 15*s*.

(30) If a gentleman spend £152. 10*s*. every week; what must be his daily income that in 15 years he may lay by £7522. 10*s*.?

Answer: £23. 2*s*.

68. Questions frequently occur, in which it is necessary to repeat the process just explained, and they are on this account said to belong to the *Double Rule of Three*: but we shall here adapt what has already been done, to the solution of a single example, which will be sufficient to point out the steps to be pursued in every other instance.

Ex. If a person travel 300 miles in 10 days, when the day is 12 hours long; how many days will it take him to travel 600 miles, when the day is 15 hours long?

We will here give two solutions, each of which produces the same result.

First Solution.

mi. mi. days.
300 : 600 :: 10
10
3,00)60,00
20 days, in which he will travel 600 miles, when the days are 12 hours long:

hrs. hrs. days.
15 : 12 :: 20
20
15)240
16 days of 15 hours each, in which he will perform the same distance.

Second Solution.

hrs. hrs. days.
15 : 12 :: 10
10
15)120
8 days of 15 hours each, in which he will travel 300 miles:

mi. mi. days.
300 : 600 :: 8
8
3,00)48,00
16 days of 15 hours each, in which he will travel 600 miles.

Examples for Practice.

(1) If the expenses of 7 persons for 3 months amount to 70 guineas; what will be the expenditure of 10 persons for 12 months at the same rate?

Answer: £420.

(2) If 10 horses consume 7bush. 2pks. of oats in 7 days; in what time will 28 horses consume 3qrs. 6bush. at the same rate?

Answer: 10 days.

(3) If 10 men reap 20 acres of corn in 4 days; how many men can reap 70 acres in 10 days, at the same rate of labour?

Answer: 14 men.

(4) If 48 men can do a piece of work in 16 days of 9 hours each: in how many days of 12 hours each will 64 men be able to do a piece of work three times as great?

Answer: 27 days.

(5) If the carriage of 13cwt. 2qrs. 19lbs. for 35 miles come to £4. 17*s*. 6*d*.; what must be paid for the conveyance of 41cwt. 1lb. for 49 miles?

Answer: £20. 9*s*. 6*d*.

(6) If £20. in trade gain £16. in 15 months, what sum will gain £24. in 3 months, at the same rate?

Answer: £150.

(7) If 12 men can perform a piece of work in 20 days; required the number of men who could perform another piece of work four times as great in a fifth part of the time.

Answer: 240 men.

(8) If with a capital of £1000., a tradesman gain £100. in 7 months, in what time will he gain £60. 10*s*., with a capital of £385.?

Answer: 11 months.

CHAPTER IV.

THE DOCTRINE OF FRACTIONS,

USUALLY TERMED VULGAR FRACTIONS.

69. DEF. ALL whole numbers, or, as they are generally called, *Integers*, being supposed to be formed by the repetition of an unit, may therefore be regarded as the result of the *multiplication* of that element; but if an unit be considered capable of *division* into any number of *equal* portions, the quantities thence arising must be viewed in the light of *broken* magnitudes; and these are therefore termed *Fractions*, or more generally, *Vulgar Fractions*, in order to distinguish them from fractions of a different form, whose nature will be discussed in the next chapter.

NOTATION, &c. OF FRACTIONS.

70. DEF. 1. If we suppose the *unit* to be divided into 2, 3, 4, 5, &c., equal portions, *one* of the portions in each case is represented by $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, $\frac{1}{5}$, &c., which may be regarded as the *primitive Fractions* of their respective denominations, and are called the *Reciprocals* of the natural numbers 2, 3, 4, 5, &c.: also, the fractions $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, $\frac{1}{5}$, &c., are read *one-half*, *one-third*, *one-fourth*, *one-fifth*, &c.

71. DEF. 2. If *two* or *more* of the equal portions into which an unit is supposed capable of being divided, be taken together, the *aggregates* thence arising are expressed by repeating the unit as *often* as such portions are repeated, the number below the line remaining the same.

Thus, if the primitive fraction $\frac{1}{3}$ be taken *twice*, there will arise a new fraction expressed by $\frac{2}{3}$: if $\frac{1}{4}$ be repeated *thrice*, there results a new fraction expressed by $\frac{3}{4}$: again, if $\frac{1}{5}$ be taken *four times*, the new fraction corresponding will be $\frac{4}{5}$; and similarly of all the other primitive fractions: also, the fractions $\frac{2}{3}$, $\frac{3}{4}$, $\frac{4}{5}$, &c., are read *two-thirds*, *three-fourths*, *four-fifths*, &c.: and all quantities of this form are called *Simple Fractions*.

72. Def. 3. Hence, in every simple fraction, the number *below* the line denotes the number of equal portions into which the unit is supposed to be divided, and is therefore called the *Denominator;* and the number *above* the line, expressing the number of such equal portions intended to be taken, is therefore termed the *Numerator.*

Thus, of the fraction $\frac{5}{7}$, whose *Terms* are 5 and 7, the denominator 7 below the line implies that the unit is supposed to be divided into *seven* equal portions; and the numerator 5 above it shews that *five* of such equal portions are here the object of our consideration: and hence it is also manifest, that the *integer* 5 is 7 times as great as the *fraction* $\frac{5}{7}$; and 5 may therefore be expressed in a fractional form by $\frac{5}{1}$.

73. From the last article it follows, that if the numerator be less than the denominator, the value of the fraction is less than unity; if the numerator be equal to the denominator, the value of the fraction is unity, and if the numerator be greater than the denominator, the value of the fraction is greater than unity.

74. Def. 4. If the numerator be less than the denominator, the fraction is termed a *Proper Fraction;* but if the numerator be greater than the denominator, it is called an *Improper Fraction:* also, if these two terms be equal to one another, we have merely the representation of the unit in a fractional form.

Thus, $\frac{2}{5}$ is a proper fraction, $\frac{11}{6}$ an improper fraction, and $\frac{7}{7}$ is merely a representation of the unit in a fractional form, being of the same value as $\frac{8}{8}$, $\frac{9}{9}$, &c.

75. From the preceding view of fractions, we are enabled to find those which arise from their multiplication and division by an integer.

If the fraction $\frac{4}{13}$ be multiplied by the integer 3, the product is evidently $\frac{4 \times 3}{13} = \frac{12}{13}$; because in $\frac{12}{13}$, *three* times as many parts of the unit are implied, as there are in $\frac{4}{13}$.

If the fraction $\frac{2}{7}$ be divided by 3, the quotient will be $\frac{2}{7 \times 3} = \frac{2}{21}$; because the same numbers of parts are

taken in $\frac{2}{7}$ and $\frac{2}{21}$, and each part in the former is *three* times as great as each part in the latter, by (72).

Hence, to *multiply* and *divide* a fraction by a whole number, we have only to multiply the *numerator* and *denominator* by it, respectively.

76. The same kind of reasoning will enable us to represent what is called a *Compound Fraction* in the form of a simple one.

A *Compound Fraction* is made up of two or more simple fractions connected together by the word *of*, as for instance, $\frac{1}{3}$ of $\frac{4}{5}$ of $\frac{6}{7}$:

$$\left.\begin{array}{l} \text{now, } \dfrac{1}{5} \text{ of } \dfrac{6}{7} = \dfrac{6}{7} \div 5 = \dfrac{6}{35} \\[2ex] \text{and } \dfrac{4}{5} \text{ of } \dfrac{6}{7} = \dfrac{6}{35} \times 4 = \dfrac{24}{35} \end{array}\right\} \text{by the last article:}$$

whence, $\frac{1}{3}$ of $\frac{4}{5}$ of $\frac{6}{7}$ is evidently the same as

$$\frac{1}{3} \text{ of } \frac{24}{35} = \frac{24}{35} \div 3 = \frac{24}{105},$$

which is a simple fraction of the ordinary form: that is,

$$\frac{1}{3} \text{ of } \frac{4}{5} \text{ of } \frac{6}{7} = \frac{1 \times 4 \times 6}{3 \times 5 \times 7} = \frac{24}{105}:$$

and from this, we infer that a compound fraction is equivalent to the simple fraction formed by multiplying together respectively the numerators and the denominators of its constituent simple fractions.

TRANSFORMATION OF FRACTIONS.

77. *If the numerator and denominator of a fraction be both multiplied or divided by the same number, the value of the fraction will not be altered.*

For, if the fraction $\frac{3}{7}$ be multiplied by 5, the product is $\frac{15}{7}$: and again if this be divided by 5, the quotient is $\frac{15}{35}$, by the last article but one: but since these two operations are the reverse of, and therefore neutralize, each other, it follows that

$$\frac{3}{7} = \frac{15}{35} = \frac{3 \times 5}{7 \times 5};$$

and also, that

$$\frac{15}{35}=\frac{3}{7}=\frac{15\div 5}{35\div 5}.$$

By means of this article, a whole number may be expressed in the form of a fraction with any denominator we please: thus,

transformation of Fractions

$$5=\frac{5}{1}=\frac{10}{2}=\frac{15}{3}=\frac{20}{4}=\&c.$$

Also, a fraction may be transformed into another with a *given* denominator, provided it be a *multiple* of the denominator of the proposed fraction: thus, $\frac{7}{8}$ may be transformed so as to have 96 for its denominator, because

$$\frac{7}{8}=\frac{7\times 12}{8\times 12}=\frac{84}{96}.$$

78. Since

$$\frac{5}{8}\times 4=\frac{20}{8}=\frac{5\times 4}{2\times 4}=\frac{5}{2};$$

for the *Multiplication* of a fraction by an integer, it appears to be immaterial whether the numerator be multiplied, or the denominator be divided, by it: and inasmuch as

$$\frac{8}{9}\div 4=\frac{8}{36}=\frac{2\times 4}{9\times 4}=\frac{2}{9};$$

for the *Division* of a fraction by a whole number, it amounts to the same thing whether we divide the numerator, or multiply the denominator, by it.

79. *A quantity made up of two others, one of which is an integer and the other a fraction, may be represented in the form of a fraction alone.*

Let us take $3\frac{4}{5}$, which is called a *mixed* quantity, and is intended to express the integer 3 and the fraction $\frac{4}{5}$ taken together, and must be read *three and four-fifths*: then, since

$$3=\frac{3}{1}=\frac{15}{5},$$

the mixed quantity $3\frac{4}{5}$ is equivalent to $\frac{15}{5}$ and $\frac{4}{5}$ taken together, or, to $\frac{19}{5}$ by the second definition: and this operation put in the form,

$$3\tfrac{4}{5}=\frac{3\times 5+4}{5}=\frac{15+4}{5}=\frac{19}{5},$$

gives the following rule.

Transform a fraction: multiply or divide the numerator & denominator by the same quantity whatever it be.

RULE. Multiply the integer by the denominator of the fraction: to the product add the numerator, and the result will be the required numerator, which placed over the denominator will form the *improper* fraction required.

Examples for Practice.

(1) Express $2\frac{5}{7}$, $5\frac{4}{9}$, $12\frac{7}{12}$ and $54\frac{8}{11}$ in the forms of improper fractions.

$$\text{Answers: } \frac{19}{7}, \frac{49}{9}, \frac{151}{12} \text{ and } \frac{602}{11}.$$

(2) Reduce to fractional forms, the mixed quantities, $41\frac{7}{13}$, $123\frac{4}{17}$, $275\frac{14}{15}$ and $374\frac{54}{103}$.

$$\text{Answers: } \frac{540}{13}, \frac{2095}{17}, \frac{4139}{15} \text{ and } \frac{38576}{103}.$$

80. A compound fraction formed of mixed quantities, may therefore by the last article be exhibited in the form of a simple fraction: thus,

$$2\tfrac{2}{3} \text{ of } 5\tfrac{1}{6} = \frac{8}{3} \text{ of } \frac{31}{6} = \frac{8 \times 31}{3 \times 6} = \frac{248}{18} = \frac{124}{9}.$$

Examples for Practice.

Exhibit the compound fractions, $13\frac{5}{9}$ of $7\frac{1}{3}$; $\frac{3}{4}$ of $\frac{4}{9}$ of $12\frac{1}{2}$, and $15\frac{7}{11}$ of $8\frac{4}{5}$ of $13\frac{2}{7}$, as improper fractions.

$$\text{Answers: } \frac{2684}{27}, \frac{25}{6} \text{ and } \frac{64672}{35}.$$

81. By means of the preceding articles, what is called a *Complex Fraction* may be reduced to a simple one; thus,

$$\frac{2\frac{1}{5}}{3\frac{2}{9}} = \frac{\frac{11}{5}}{\frac{29}{9}} = \frac{\frac{11}{5} \times 5 \times 9}{\frac{29}{9} \times 9 \times 5} = \frac{11 \times 9}{29 \times 5} = \frac{99}{145},$$

a simple fraction, obtained by multiplying both the compound numerator and denominator by the product of the denominators of their fractional parts.

82. *A quantity in the form of an improper fraction may always be expressed by a mixed quantity.*

We see immediately that $\frac{35}{8}$ is equivalent to $\frac{32+3}{8}$, or, to $\frac{32}{8}$ and $\frac{3}{8}$ taken together: but $\frac{32}{8}$ is equal to the

integer 4, and therefore the required mixed quantity will be equal to the integer 4 and the proper fraction $\frac{3}{8}$ taken together, which is sometimes expressed by $4 + \dfrac{3}{8}$, but more generally in the form $4\frac{3}{8}$.

This process is evidently the same thing as dividing both the numerator and denominator by the *denominator*, and noticing the remainder of the former: and stated in the form,

$$8\,\underline{)\,3\,5\,}$$
$$\underline{4\tfrac{3}{8}}$$

it suggests the following rule.

RULE. Divide the numerator of the fraction by the denominator, and the quotient will be the integral part; and the fractional part will be formed by making the remainder the numerator of a fraction having the same denominator as the one proposed. If there be no remainder, the fraction is equivalent to the integer thus found.

Examples for Practice.

(1) Find the mixed quantities equivalent to

$$\frac{19}{5},\ \frac{38}{9},\ \frac{149}{11} \text{ and } \frac{199}{12}.$$

Answers: $3\frac{4}{5}$, $4\frac{2}{9}$, $13\frac{6}{11}$ and $16\frac{7}{12}$.

(2) Express $\dfrac{440}{13}$, $\dfrac{2417}{19}$, $\dfrac{3797}{29}$ and $\dfrac{30471}{37}$, as mixed quantities.

Answers: $33\frac{11}{13}$, $127\frac{4}{19}$, $130\frac{27}{29}$ and $823\frac{20}{37}$.

(3) Represent the following fractional quantities,

$$\frac{8357}{278},\ \frac{18793}{359},\ \frac{1}{3} \text{ of } \frac{4}{7} \text{ of } 6\tfrac{3}{8}, \text{ and } \frac{5}{6} \text{ of } \frac{13\frac{2}{5}}{4\frac{7}{9}},$$

in the forms of mixed quantities.

Answers: $30\frac{17}{278}$, $52\frac{125}{359}$, $1\frac{3}{14}$ and $2\frac{173}{215}$.

83. *A fraction may be reduced to its lowest terms, by dividing both its numerator and denominator, by their greatest common measure.*

For, since the value of a fraction is not altered by dividing its numerator and denominator by any factor

common to them both, it will necessarily be expressed in its *lowest* or *simplest* terms, when that factor is the *greatest* common measure, determined by the Rule of Article (53).

If the greatest common measure be 1, the numerator and denominator are *prime* to each other, and the fraction is already in its lowest terms.

Ex. Reduce the fraction $\frac{825}{960}$ to its lowest terms.

By Article (53) above mentioned, we have

$$
\begin{array}{l}
825)\,960\,(1 \\
\quad\;\; 825 \\
\quad\;\; \overline{135)\,825\,(6} \\
\qquad\qquad 810 \\
\qquad\qquad \overline{\;\;15)\,135\,(9} \\
\qquad\qquad\qquad\; 135 \\
\qquad\qquad\qquad\; \overline{}
\end{array}
$$

and 15 is therefore the greatest common measure: and dividing each of the terms of the fraction by it, as follows:

$$
\begin{array}{l|l}
15)\,825\,(55, \quad & \quad 15)\,960\,(64, \\
\quad\;\; 75 & \quad\quad\;\; 90 \\
\quad\;\; \overline{\;\;75} & \quad\quad\;\; \overline{\;\;60} \\
\quad\;\;\;\; \underline{75} & \quad\quad\;\;\;\; \underline{60}
\end{array}
$$

we have $\frac{55}{64}$ for the *equivalent* fraction expressed in the least terms possible.

The terms of the *original* fraction are equal multiples, or *equimultiples*, of those of the equivalent *reduced* one.

84. In many instances it is unnecessary to find the greatest common measure at first, the fractions being reducible to lower terms by successive divisions of the numerators and denominators by common factors discovered by *inspection*.

$$\text{Thus, } \frac{4968}{5904}=\frac{2484}{2952}=\frac{1242}{1476}=\frac{621}{738}=\frac{207}{246}=\frac{69}{82},$$

from *three* successive divisions of the numerator and denominator by 2, and then from *two* successive divisions

by 3: and these are the terms which would have been obtained from dividing at once by 72, the greatest common measure found by the rule.

Examples for Practice.

(1) Reduce $\frac{9}{24}$, $\frac{63}{144}$, $\frac{147}{189}$ and $\frac{435}{957}$ to their lowest terms.

Answers: $\frac{3}{8}$, $\frac{7}{16}$, $\frac{7}{9}$ and $\frac{5}{11}$.

(2) Express in their simplest forms, the fractions, $\frac{3094}{3042}$, $\frac{3444}{3556}$, $\frac{5565}{8533}$ and $\frac{7568}{9504}$.

Answers: $\frac{119}{117}$, $\frac{123}{127}$, $\frac{15}{23}$ and $\frac{43}{54}$.

(3) Find the simplest fractions expressive of the values of $\frac{13667}{14186}$, $\frac{13478}{16701}$, $\frac{43365}{44688}$ and $\frac{48510}{49005}$.

Answers: $\frac{79}{82}$, $\frac{46}{57}$, $\frac{295}{304}$ and $\frac{98}{99}$.

(4) Reduce as much as possible, the fractions, $\frac{8398}{29393}$, $\frac{11050}{35581}$, $\frac{109375}{10000000}$ and $\frac{135795}{222210}$.

Answers: $\frac{2}{7}$, $\frac{50}{161}$, $\frac{7}{640}$ and $\frac{11}{18}$.

85. *Two or more fractions having different denominators, may be transformed into other equivalent fractions having a common denominator.*

Let it be required to reduce $\frac{1}{2}$, $\frac{2}{5}$ and $\frac{3}{7}$ to a common denominator; then, since the continued product of the denominators is expressed by $2 \times 5 \times 7$, we have

$$\frac{1}{2} = \frac{1 \times 5 \times 7}{2 \times 5 \times 7} = \frac{35}{70}:$$

$$\frac{2}{5} = \frac{2 \times 2 \times 7}{5 \times 2 \times 7} = \frac{28}{70}:$$

$$\frac{3}{7} = \frac{3 \times 2 \times 5}{7 \times 2 \times 5} = \frac{30}{70}:$$

so that $\frac{35}{70}$, $\frac{28}{70}$ and $\frac{30}{70}$ are the new *equivalent* fractions with the common denominator 70; and the steps taken manifestly lead to the same thing as the operations here subjoined:

$$\text{first,}\quad \left.\begin{aligned} 1\times5\times7&=35\\ 2\times2\times7&=28\\ 3\times2\times5&=30 \end{aligned}\right\}\ \text{new numerators:}$$

$$\text{and } 2\times5\times7=70, \text{ common denominator:}$$

wherefore the new equivalent fractions are $\frac{35}{70}$, $\frac{28}{70}$ and $\frac{30}{70}$, as above: and hence we derive the following rule.

Rule. Multiply each numerator by all the denominators *except* the one placed under it, and the product will be the corresponding new numerator: and multiply together the denominators of all the fractions for a common denominator.

86. If two or more of the denominators have a common measure, the equivalent fractions may be expressed in simpler terms than obtainable by the Rule, and still having a common denominator: thus, if the fractions be $\frac{1}{2}$, $\frac{2}{3}$ and $\frac{3}{4}$, we have from Article (56),

$$\frac{2\times3}{1}=6, \text{ and } \frac{6\times4}{2}=12,$$

the least common multiple of the denominators: also,

$$\left.\begin{aligned} \frac{1}{2}&=\frac{1\times6}{2\times6}=\frac{6}{12}\\ \frac{2}{3}&=\frac{2\times4}{3\times4}=\frac{8}{12}\\ \frac{3}{4}&=\frac{3\times3}{4\times3}=\frac{9}{12} \end{aligned}\right\}\ \text{the equivalent fractions,}$$

with the *least* common denominator 12; and the new numerators are here obtained by multiplying those of the fractions proposed by the quotients arising from its division by their respective denominators.

It need scarcely be observed that mixed quantities, compound and complex fractions, must all be reduced to the forms of simple fractions, before this and the subsequent rules can be applied: and that the magnitudes of fractional quantities may also be compared with each other by what is here done.

Examples for Practice.

(1) Reduce $\frac{2}{3}$ and $\frac{4}{5}$; $\frac{1}{7}$ and $\frac{2}{9}$; $\frac{3}{4}$ and $\frac{9}{11}$ respectively, to common denominators.

$$\text{Answers: } \frac{10}{15}, \frac{12}{15}; \frac{9}{63}, \frac{14}{63}, \text{ and } \frac{33}{44}, \frac{36}{44}.$$

(2) Reduce to common denominators, $\frac{1}{2}$, $\frac{4}{5}$ and $\frac{6}{7}$; also, $\frac{2}{3}$, $\frac{3}{7}$ and $\frac{4}{11}$.

$$\text{Answers: } \frac{35}{70}, \frac{56}{70}, \frac{60}{70} \text{ and } \frac{154}{231}, \frac{99}{231}, \frac{84}{231}.$$

(3) Reduce $\frac{4}{5}$, $2\frac{5}{9}$ and $3\frac{2}{11}$ to fractions, having a common denominator.

$$\text{Answer: } \frac{396}{495}, \frac{1265}{495} \text{ and } \frac{1575}{495}.$$

(4) Transform $\frac{2}{3}$, $\frac{3}{4}$ and $\frac{7}{8}$ into equivalent fractions, with the least common denominator.

$$\text{Answer: } \frac{16}{24}, \frac{18}{24} \text{ and } \frac{21}{24}.$$

(5) Reduce $\frac{1}{12}$, $\frac{1}{16}$, $\frac{1}{21}$ and $\frac{1}{60}$, so as to have the least common denominator.

$$\text{Answer: } \frac{140}{1680}, \frac{105}{1680}, \frac{80}{1680} \text{ and } \frac{28}{1680}.$$

(6) Reduce $\frac{1}{3}$, $\frac{2}{9}$, $\frac{5}{12}$, and $\frac{7}{18}$ to the least common denominator.

$$\text{Answer: } \frac{12}{36}, \frac{8}{36}, \frac{15}{36} \text{ and } \frac{14}{36}.$$

(7) Express with the least common denominator, $\frac{1}{2}$, $\frac{2}{3}$, $\frac{3}{4}$, $\frac{4}{5}$ and $\frac{5}{6}$.

$$\text{Answer: } \frac{30}{60}, \frac{40}{60}, \frac{45}{60}, \frac{48}{60} \text{ and } \frac{50}{60}.$$

(8) Compare the quantities $2\frac{1}{2}$, $\frac{2}{7}$ of $9\frac{2}{5}$ and $\dfrac{7\frac{1}{2}}{2\frac{5}{6}}$.

$$\text{Answer: } \frac{2975}{1190}, \frac{3196}{1190} \text{ and } \frac{3150}{1190}.$$

I. ADDITION OF FRACTIONS.

RULE. Reduce the proposed quantities, if need be, to equivalent fractions with a common denominator; add together the new numerators, and under their sum place the common denominator: and the resulting fraction, reduced when possible, will be the sum required.

For, let $7\frac{2}{7}$ and $4\frac{7}{8}$ be the proposed quantities, which reduced to improper fractions are $\frac{51}{7}$ and $\frac{39}{8}$: then, since addition can be performed only upon quantities of the *same* denominations, these fractions must first be reduced to a *common* denominator; and their sum will be

$$\frac{51}{7} + \frac{39}{8} = \frac{408}{56} + \frac{273}{56} = \frac{681}{56} = 12\tfrac{9}{56}.$$

This process may be rendered simpler as follows: for, the sum of the integers $= 7 + 4 = 11$:

the sum of the fractions $= \frac{2}{7} + \frac{7}{8} = \frac{16 + 49}{56} = \frac{65}{56} = 1\frac{9}{56}$:

and therefore the entire sum $= 11 + 1\frac{9}{56} = 12\frac{9}{56}$,

as before; and this is much shorter and easier, particularly when the numbers are large: also, each of these methods is evidently applicable, whatever be the number of quantities proposed.

Examples for Practice.

(1) Find the sums of $\frac{2}{5}$ and $\frac{4}{7}$; of $\frac{3}{7}$ and $\frac{5}{9}$; of $\frac{3}{8}$ and $\frac{7}{9}$, and of $\frac{5}{8}$ and $\frac{12}{17}$.

Answers: $\frac{34}{35}$, $\frac{62}{63}$, $1\frac{11}{72}$ and $1\frac{45}{136}$.

(2) Add together $1\frac{1}{2}$ and $7\frac{1}{5}$; $2\frac{6}{7}$ and $13\frac{3}{10}$; $5\frac{1}{6}$ and $12\frac{4}{5}$, and $37\frac{2}{11}$ and $24\frac{12}{13}$.

Answers: $8\frac{7}{10}$, $16\frac{11}{70}$, $17\frac{29}{30}$ and $62\frac{15}{143}$.

(3) What are the sums of $\frac{41}{18}$ and $\frac{21}{13}$; of $\frac{57}{5}$ and $\frac{38}{15}$; of $\frac{31}{9}$ and $\frac{49}{8}$, and of $\frac{27}{16}$ and $\frac{71}{24}$.

Answers: $3\frac{209}{234}$, $13\frac{14}{15}$, $9\frac{41}{72}$ and $4\frac{31}{48}$.

(4) Add together $\frac{2}{3}$, $\frac{3}{4}$ and $\frac{5}{6}$; $\frac{1}{7}$, $\frac{2}{5}$ and $\frac{3}{11}$, and $\frac{4}{5}$, $\frac{6}{7}$ and $\frac{9}{10}$.

Answers: $2\frac{1}{4}$, $\frac{314}{385}$ and $2\frac{39}{70}$.

(5) Add together $\frac{11}{16}$, $\frac{45}{8}$, and $\frac{97}{2}$; $2\frac{1}{5}$, $3\frac{2}{9}$ and $5\frac{2}{8}$, and $8\frac{4}{7}$, $13\frac{5}{9}$ and $27\frac{8}{11}$.

Answers: $54\frac{13}{16}$, $10\frac{287}{360}$ and $49\frac{592}{693}$.

(6) Find the respective sums of $\frac{2}{3}$, $\frac{3}{4}$, $\frac{5}{6}$ and $\frac{7}{8}$: of $\frac{1}{4}$, $\frac{2}{5}$, $\frac{3}{7}$ and $\frac{4}{9}$, and of $\frac{1}{40}$, $\frac{7}{20}$, $\frac{6}{5}$ and $\frac{1}{8}$.

Answers: $3\frac{1}{8}$, $1\frac{659}{1260}$ and $1\frac{7}{10}$.

(7) Find the respective sums of $1\frac{2}{7}$, $\frac{5}{9}$, $\frac{8}{11}$ and $3\frac{1}{5}$: and of $3\frac{5}{8}$, $2\frac{3}{5}$, $\frac{7}{11}$ and $7\frac{1}{6}$.

Answers: $5\frac{2668}{3465}$ and $14\frac{37}{1980}$.

(8) Add together $\frac{2}{5}$, $\frac{35}{80}$, $\frac{14}{100}$, $\frac{3}{140}$ and $\frac{3}{2800}$: also, $387\frac{1}{2}$, $285\frac{1}{4}$, $394\frac{1}{3}$ and $\frac{2}{5}$ of 3704.

Answers: 1 and $2548\frac{41}{60}$. $=14\frac{3}{4}+\frac{2}{3}\times\frac{5}{6}$

(9) Required the respective sums of $14\frac{3}{4}$ and $\frac{2}{3}$ of $\frac{5}{6}$ of 8: of $\frac{2}{7}$, $4\frac{1}{3}$ and $\frac{3}{5}$ of 2: and of $\frac{2}{3}$ of $\frac{5}{7}$, 9, $\dfrac{2\frac{4}{5}}{7}$ and $\dfrac{1\frac{2}{8}}{2\frac{1}{2}}$.

Answers: $19\frac{7}{36}$, $5\frac{86}{105}$ and $10\frac{19}{35}$.

(10) Express the value of $1\frac{1}{3}+\dfrac{8}{3}$ of $\dfrac{41}{34}\div\dfrac{4}{5\frac{1}{10}}$, by a fraction in its lowest terms.

Answer: $\dfrac{16}{3}$.

II. SUBTRACTION OF FRACTIONS.

RULE. Transform the proposed quantities, if necessary, so as to have a common denominator; subtract the less numerator from the greater; under the remainder place the common denominator, and the result properly reduced, will be the required difference.

For, taking the quantities $5\frac{1}{3}$ and $1\frac{4}{7}$, and reducing them to fractional forms, we have, for the reason mentioned in the last rule, the difference

$$=\frac{16}{3}-\frac{11}{7}=\frac{112}{21}-\frac{33}{21}=\frac{79}{21}=3\tfrac{16}{21}.$$

Like the last, this operation may frequently be performed in a more convenient form as follows: the difference $=5\frac{1}{3}-1\frac{4}{7}=5\frac{7}{21}-1\frac{12}{21}=3\frac{16}{21}$: where $\frac{12}{21}$, being

greater than $\frac{7}{21}$, is subtracted from $\dfrac{7}{21}+1$ or $\dfrac{28}{21}$, and 1 is carried to the whole number 1, as in the Subtraction of Integers.

Examples for Practice.

(1) Find the differences of $\frac{3}{5}$ and $\frac{1}{6}$; of $\frac{7}{9}$ and $\frac{3}{7}$; of $\frac{2}{9}$ and $\frac{3}{11}$, and of $\frac{5}{6}$ and $\frac{8}{15}$.

Answers: $\dfrac{13}{30}$, $\dfrac{22}{63}$, $\dfrac{5}{99}$ and $\dfrac{3}{10}$.

(2) What are the respective differences of $19\frac{2}{7}$ and $13\frac{2}{11}$; of $8\frac{12}{25}$ and $17\frac{13}{27}$, and of 1000 and $384\frac{7}{39}$?

Answers: $6\frac{8}{77}$, $9\frac{1}{675}$ and $615\frac{32}{39}$.

(3) Required the difference of $1\frac{2}{3}$ of $3\frac{5}{8}$ and $2\frac{7}{9}$ of $16\frac{2}{7}$; also, of $\frac{2}{5}$ of $\frac{3}{8}$ of $\frac{5}{9}$ and $\frac{3}{7}$ of $\frac{2}{11}$ of 25.

Answers: $39\frac{11}{56}$ and $1\frac{799}{924}$.

(4) Find the difference of $\dfrac{3}{5}$ of $\dfrac{4\frac{1}{6}}{5\frac{1}{7}}$ and $\dfrac{2}{3}$ of $\dfrac{15}{2}$: also, of $\dfrac{3\frac{2}{5}}{4\frac{7}{9}}$ and $\dfrac{6\frac{2}{8}}{12\frac{2}{7}}$.

Answers: $4\frac{87}{72}$ and $\frac{409}{645}$.

(5) Prove that the sum of $5\frac{1}{3}$ and $3\frac{1}{5}$, is equal to four times their difference.

III. MULTIPLICATION OF FRACTIONS.

Rule. Multiply together the respective numerators and denominators of the proposed quantities, reduced to fractional forms if necessary; and the fraction thence arising will be the product, which may generally be simplified by means of the preceding articles.

For, let the fractions be $\frac{2}{9}$ and $\frac{7}{8}$; then if $\frac{2}{9}$ be multiplied by 7, the product will be $\frac{14}{9}$ by article (75): but 7 being 8 times as great as $\frac{7}{8}$, the multiplier above used is 8 times too large, and the product $\frac{14}{9}$ will therefore be 8 times too large also: whence the product required must be

$$\frac{14}{9} \div 8 = \frac{14}{72} = \frac{7}{36}:$$

that is,

$$\text{the product} = \frac{2}{9} \times \frac{7}{8} = \frac{2 \times 7}{9 \times 8} = \frac{14}{72} = \frac{7}{36}.$$

87. If three or more quantities be proposed, as $\frac{1}{2}$, $\frac{2}{3}$ and $\frac{3}{4}$; their continued product is $\frac{1}{2} \times \frac{2}{3} \times \frac{3}{4}$

$$= \frac{1}{2} \times \text{the product of } \frac{2}{3} \text{ and } \frac{3}{4} = \frac{1}{2} \times \frac{6}{12} = \frac{6}{24} = \frac{1}{4};$$

$$\text{or, } \frac{1}{2} \times \frac{2}{3} \times \frac{3}{4} = \frac{1 \times 2 \times 3}{2 \times 3 \times 4} = \frac{6}{24} = \frac{1}{4};$$

and thus the rule may be proved to be general: also, in cases like this, the reduction is much shortened by *cancelling* from the products of the numerators and denominators, any factor or factors common to them both, and effecting the multiplications of what are left; as,

$$\frac{1}{2} \times \frac{2}{3} \times \frac{3}{4} = \frac{1 \times 2 \times 3}{2 \times 3 \times 4} = \frac{1 \times 1 \times 1}{1 \times 1 \times 4} = \frac{1}{4},$$

the product before found.

Examples for Practice.

(1) Required the respective products of $\frac{2}{5}$ and $\frac{3}{7}$; of $\frac{3}{8}$ and $\frac{5}{9}$; of $2\frac{3}{5}$ and $7\frac{2}{9}$, and of $8\frac{2}{7}$ and $10\frac{5}{11}$.

Answers: $\frac{6}{35}$, $\frac{5}{24}$, $18\frac{7}{9}$ and $86\frac{48}{77}$.

(2) Find the continued products of $\frac{2}{3}$, $\frac{3}{5}$ and $\frac{7}{12}$: of $\frac{3}{8}$, $\frac{11}{7}$ and $\frac{15}{11}$: of $\frac{49}{133}$, $1\frac{1}{75}$ and $\frac{28}{98}$, and of $\frac{428}{515}$, $\frac{5253}{1819}$ and $\frac{5}{4}$.

Answers: $\frac{7}{30}$, $\frac{45}{56}$, $\frac{8}{75}$ and 3.

(3) Required the continued products of $\frac{1}{2}$, $\frac{2}{3}$, $\frac{3}{4}$ and $\frac{4}{5}$: of $\frac{3}{4}$, $\frac{6}{7}$, $\frac{8}{9}$ and $\frac{10}{11}$, and of $\frac{8}{13}$, $2\frac{2}{3}$, $1\frac{1}{25}$ and $1\frac{11}{64}$.

Answers: $\frac{1}{5}$, $\frac{40}{77}$ and 2.

(4) Multiply $2\frac{3}{8}$ by $\frac{1}{3}$ of $\frac{2}{5}$ of $\frac{7}{9}$, and $13\frac{5}{9}$ of $7\frac{1}{3}$ by $\frac{3}{4}$ of $\frac{4}{9}$ of $12\frac{1}{2}$.

Answers: $\frac{133}{540}$ and $414\frac{16}{81}$.

(5) Multiply together $\frac{2\frac{3}{4}}{5\frac{7}{9}}$ of $\frac{1}{3}$ and $\frac{3}{5}$ of $\frac{4\frac{3}{8}}{7\frac{1}{2}}$.

Answer: $\frac{231}{4160}$.

(6) Find the continued product of the fractions, $\frac{324}{361}$, $\frac{1444}{1296}$, $\frac{441}{529}$ and $\frac{2116}{1764}$.

Answer: 1.

IV. DIVISION OF FRACTIONS.

RULE. Multiply the dividend by the divisor *inverted*, and this result reduced when possible, will be the quotient: or, which is the same thing, *invert* the divisor, and then proceed according to the rule for the Multiplication of Fractions.

For, let $\frac{3}{7}$ be to be divided by $\frac{4}{5}$; then it is manifest that $\frac{3}{7} \div 4 = \frac{3}{28}$ is 5 times too *small*, because the divisor has been taken 5 times too *great*: whence the quotient required will be

$$\frac{3}{28} \times 5 = \frac{15}{28}:$$

that is, the quotient is

$$\frac{15}{28} = \frac{3 \times 5}{7 \times 4} = \frac{3}{7} \times \frac{5}{4};$$

and the operation may be expressed in this form;

$$\text{the quotient} = \frac{3}{7} \div \frac{4}{5} = \frac{3}{7} \times \frac{5}{4} = \frac{15}{28}.$$

88. To denote the division of one integer by another, as for instance, that of 4 by 5, we shall have, according to the principles already established,

$$\text{the quotient} = \frac{4}{1} \div \frac{5}{1} = \frac{4}{1} \times \frac{1}{5} = \frac{4}{5};$$

or, in words, a simple fraction may be considered as an adequate expression of the implied division of its numerator by its denominator.

Examples for Practice.

(1) Find the respective quotients of $\frac{2}{7}$ by $\frac{3}{8}$; of $\frac{3}{5}$ by $\frac{4}{9}$; of $\frac{4}{11}$ by $\frac{6}{13}$, and of $\frac{19}{5}$ by $\frac{83}{10}$.

Answers: $\frac{16}{21}$, $1\frac{7}{20}$, $\frac{26}{33}$ and $\frac{38}{83}$.

(2) What are the respective quotients of $2\frac{1}{5}$ by $3\frac{2}{9}$; of $10\frac{4}{9}$ by $13\frac{3}{7}$, and of $17\frac{2}{3}$ by $7\frac{13}{14}$?

Answers: $\frac{99}{145}$, $\frac{7}{9}$ and $2\frac{76}{333}$.

(3) Divide $3\frac{4}{9}$ by $\frac{1}{2}$ of $\frac{4}{7}$ of $\frac{5}{6}$; and $15\frac{7}{11}$ of $8\frac{4}{5}$ by $\frac{4}{5}$ of $\frac{6}{11}$ of $15\frac{5}{6}$.

Answers: $14\frac{7}{15}$ and $19\frac{87}{95}$.

(4) Compare the product and quotient of $\frac{7}{9}$ by $\frac{10}{11}$.

Answer: $\frac{700}{990}$ and $\frac{847}{990}$.

89. What has been proved in the adaptation of the four fundamental operations to fractional quantities, will furnish the means of simplifying arithmetical expressions formed by any of their combinations: thus,

(1) $$\frac{1}{2}-\frac{1}{3}+\frac{1}{4}-\frac{1}{5}=\left(\frac{1}{2}+\frac{1}{4}\right)-\left(\frac{1}{3}+\frac{1}{5}\right)$$
$$=\frac{3}{4}-\frac{8}{15}=\frac{45}{60}-\frac{32}{60}=\frac{13}{60}.$$

(2) $$\left(\frac{1}{3}+\frac{1}{5}\right)\times\left(\frac{1}{2}-\frac{1}{7}\right)=\frac{8}{15}\times\frac{5}{14}=\frac{40}{210}=\frac{4}{21}.$$

(3) $$\left(\frac{4}{7}-\frac{2}{11}\right)\div\left(\frac{5}{6}+\frac{3}{8}\right)=\frac{30}{77}\div\frac{58}{48}=\frac{30\times 48}{77\times 58}$$
$$=\frac{30\times 24}{77\times 29}=\frac{720}{2233}.$$

Examples for Practice.

(1) Required the value of $\frac{5}{6}-\frac{3}{4}+\frac{2}{3}-\frac{1}{2}$.

Answer: $\frac{1}{4}$.

(2) Reduce to its simplest form, the expression,

$$\frac{1}{2}+\frac{2}{3}-\frac{1}{6}+\frac{3}{8}-\frac{1}{12}.$$

Answer: $1\frac{7}{24}$.

(3) Find the simplest fraction equivalent to

$$\frac{323852}{1640625}+\frac{1}{546875}-\frac{1}{15625}+\frac{1}{375}.$$

Answer: $\frac{1}{5}$.

(4) Reduce $\left(\frac{1}{2}+\frac{1}{3}+\frac{1}{4}\right)\times\left(\frac{1}{5}\text{ of }\frac{2}{7}\right)$ to its simplest form.

Answer: $\frac{13}{210}$.

(5) Simplify as much as possible, the arithmetical expression $\left(\frac{1}{2}\times\frac{3}{8}+\frac{3}{7}\times\frac{2}{5}\right)-\left(\frac{2}{7}\times\frac{5}{9}-\frac{1}{8}\times\frac{4}{11}\right)$.

Answer: $\frac{13619}{55440}$.

(6) Determine the simple fraction which expresses the value of $\left(\frac{5}{7}\times\frac{2}{9}\times 13\tfrac{1}{2}\right)\div\left(\frac{1}{9}\times\frac{3}{7}+54\right)$.

Answer: $\frac{9}{227}$.

(7) What is the value of the expression,

$$\frac{2247}{1017}\div\frac{903}{1107}\times\frac{774}{615}\div\frac{1926}{565}?$$

Answer: 1.

(8) Required the value of the expression,

$$\frac{3}{8}\text{ of }\frac{4}{7}-\frac{2}{11}\text{ of }3\tfrac{1}{7}+\frac{5}{9}\text{ of }3\tfrac{3}{8}.$$

Answer: $1\tfrac{29}{56}$.

REDUCTION OF FRACTIONS.

90. Our attention has hitherto been confined to fractions considered *generally*, without regard to the particular species of their *units;* and it remains to apply what has been said to such *concrete* quantities as constitute the principal subjects of practical computation.

91. *A fraction may always be transformed into another, so that the value of the unit in the latter may have a specified relation to that of the unit in the former.*

RULE. *Multiply* or *divide* the fraction proposed by the numbers which connect the different denominations in order, according as the value of the unit in the required fraction, is *less* or *greater* than that of the unit in the one which is given.

For, let the proposed fraction be £$\frac{2}{7}$, where the unit is one *pound:* then if it be required to find the corresponding fraction when the unit is one *farthing,* it is manifest from what has been said in the Reduction of compound quantities, that in order to retain the same *absolute* value, we must have $20 \times 12 \times 4$ times as great a *fraction* as the original one: that is,

$$\overset{\pounds}{\frac{2}{7}} = \frac{2}{7} \times \frac{20}{1} \times \overset{\text{farthings.}}{\frac{12}{1}} \times \frac{4}{1} = \overset{\text{far.}}{\frac{1920}{7}}:$$

and the value of the unit in the latter fraction being $\frac{1}{960}$th part of that in the former, the same *absolute value* is retained by taking 960 times as many parts in the latter, as in the former.

Again, reversing the operation, we shall have

$$\overset{\text{far.}}{\frac{1920}{7}} = \frac{1920}{7} \times \overset{\pounds.}{\frac{1}{4}} \times \frac{1}{12} \times \frac{1}{20} = \overset{\pounds.}{\frac{1920}{6720}} = \overset{\pounds.}{\frac{2}{7}};$$

the divisors 4, 12 and 20 being *inverted,* according to the rule laid down for the Division of Fractions.

Ex. Let it be required to find what fraction of a crown, is equivalent to $\frac{1}{4}$ of a pound.

According to the rule just given, we have

$$\overset{\pounds.}{\frac{1}{4}} = \overset{s.}{\frac{1}{4}} \times \frac{20}{1} = \overset{s.}{\frac{20}{4}} = \overset{\text{crowns.}}{\frac{20}{4}} \times \frac{1}{5} = \overset{\text{cr.}}{\frac{20}{20}} = \overset{\text{cr.}}{\frac{1}{1}}:$$

and we know very well that $\frac{1}{4}$ of £1, or 5*s.*, is equal to 1 crown, expressed fractionally by $\frac{1}{1}$.

Examples for Practice.

(1) Reduce $\frac{1}{7}$, $\frac{4}{9}$ and $\frac{3}{64}$ of a pound, to fractions of a penny.

Answers: $\frac{240}{7}$, $\frac{320}{3}$ and $\frac{45}{4}$.

$\frac{2}{5} \times \frac{1}{20} = \frac{1}{50}$ $\frac{5}{7} \times \frac{1}{20} \times \frac{1}{12} = \frac{5}{1680} = \frac{1}{336}$

 $\frac{160}{11} \times \frac{1}{20} \times \frac{1}{12} \times \frac{1}{4}$

(2) Express $\frac{2}{5}$ of a shilling, $\frac{5}{7}$ of a penny and $\frac{160}{11}$ of a farthing, as fractions of a pound.

Answers: $\frac{1}{50}$, $\frac{1}{336}$ and $\frac{1}{66}$.

(3) Reduce $\frac{2}{9}$ of a guinea, and $\frac{3}{4}$ of a half-guinea, to fractions of £1.

$\frac{2}{9} \times \frac{21}{20}$ $\frac{42}{180} = \frac{7}{30}$ $\frac{3}{4} \times \frac{21}{20} \times \frac{1}{2} = \frac{63}{160}$

Answers: $\frac{7}{30}$ and $\frac{63}{160}$.

(4) Reduce $\frac{3}{32}$ of a cwt. to the fraction of 1 lb, and $\frac{4}{7}$ of an ounce, to that of 1 cwt.

$\frac{3}{32} \times \frac{112}{1} = \frac{3.7.16}{2.16} = \frac{21}{2}$ $\frac{4}{7} \times \frac{1}{16} \times \frac{1}{112} = \frac{1}{3136}$

Answers: $\frac{21}{2}$ and $\frac{1}{3136}$.

feet

$\frac{7}{342} \times \frac{36}{1} = \frac{252}{342} = \frac{14}{19}$

(5) Express $\frac{7}{342}$ of a yard as the fraction of an inch, and $\frac{108}{145}$ of an inch as that of a pole.

$= \frac{108}{28710} = \frac{6}{1595}$

Answers: $\frac{14}{19}$ and $\frac{6}{1595}$.

$\times \frac{5}{4} = \frac{15}{16}$

(6) Find the fraction of a yard, which expresses $\frac{3}{4}$ of an ell of 5 quarters; and that of a day which is equivalent to $\frac{5}{146}$ of a year of 365 days. $\frac{5}{146} \times \frac{365}{1} = \frac{1825}{146} = \frac{25}{2}$

Answers: $\frac{15}{16}$ and $\frac{25}{2}$.

qts

$\frac{4}{279} \times \frac{144}{1}$ = one bar.

(7) Reduce $\frac{4}{279}$ of a barrel of beer to the fraction of a quart; and $\frac{4}{11}$ of a pint of wine, to the fraction of a hogshead.

$\frac{576}{279} = \frac{64}{31}$

$\frac{4}{11} \times \frac{1}{504}$ = hhd. ∴ $\frac{4}{11} \times \frac{1}{504} = \frac{4}{5544} = \frac{1}{1386}$

Answers: $\frac{64}{31}$ and $\frac{1}{1386}$.

£ guinea

$\frac{1}{10} \times \frac{1}{20} \times \frac{4}{7} \times \frac{21}{1}$

(8) Required the fractions of £10., which are equivalent to $\frac{4}{7}$ of a guinea, $\frac{2}{9}$ of a shilling, and $\frac{16}{15}$ of a farthing.

$\frac{84}{1400} = \frac{3}{50}$

$\frac{16}{15} \times \frac{1}{10} \times \frac{1}{20} \times \frac{1}{12} \times \frac{1}{4}$

$\times \frac{1}{10} \times \frac{1}{20} = \frac{2}{1800} = \frac{1}{900}$

Answers: $\frac{3}{50}$, $\frac{1}{900}$ and $\frac{1}{9000}$. $= \frac{16}{144000} = \frac{1}{9000}$

92. *The value of a compound quantity may be exhibited in the form of a fraction, whereof the unit is of a specified denomination.*

RULE. Reduce the proposed quantity to the lowest denomination contained in it, and also the proposed unit to the same denomination; then the fraction whose numerator and denominator are these results respectively, will be the one required.

For, let it be required to represent 2qrs. 15lbs. as the fraction of 1cwt: then we have

qrs. lbs.	cwt.
	1
2 . 15	4
28	4
71 lbs.	28
	112 lbs.

and of the 112 pounds or equal parts into which 1 cwt. is supposed to be divided, 71 are here taken, so that according to Article (72), the fraction required will be $\frac{71}{112}$ cwt.

93. By means of the two preceding rules, magnitudes of the same kind, consisting of fractions of simple or compound quantities, and connected by the operations of addition or subtraction, may be reduced to simple fractions of any given denomination.

Ex. Find the fraction of £1., which is equivalent to the excess of $\frac{2}{3}$ of a guinea, above the sum of $\frac{3}{4}$ of a shilling and $\left(\frac{4}{9}\right)$ of $7s.\ 6d.$

$\frac{4}{9}$

$$\text{Here, } \overset{g.}{\frac{2}{3}} = \frac{2}{3} \times \overset{£.}{\frac{21}{20}} = \overset{£.}{\frac{7}{10}}:$$

$$\overset{s.}{\frac{3}{4}} = \frac{3}{4} \times \frac{1}{20} = \frac{3}{80}:$$

$\frac{4}{9} \times \frac{7 \times 12}{1} + \frac{4}{1} = \frac{4}{9} \times \frac{90}{1}$

$\frac{4}{9} \times \frac{90}{1} \times \frac{1}{20} \times \frac{1}{12} = \frac{360}{2160} = \frac{1}{6}$

$$\frac{4}{9} \text{ of } 7s.\ 6d. = \frac{4}{9} \text{ of } \frac{3}{8} = \frac{1}{6}:$$

and therefore the required fraction will be

$$\frac{7}{10} - \frac{3}{80} - \frac{1}{6} = \overset{£.}{\frac{119}{240}}.$$

Examples for Practice.

(1) Express 17*s*. $11\frac{1}{2}$*d*.; 19*s*. $10\frac{3}{4}$*d*., and £1. 13*s*. $7\frac{1}{4}$*d*. $\frac{2}{7}$, as fractions of £1.

Answers: $\frac{431}{480}$, $\frac{191}{192}$ and $\left(\frac{11293}{6720}\right)$ $= \frac{1613}{3360}$

(2) What fraction is 2 cwt. 1 qr. 16 lbs. of a ton; 2 ft. 9 in. of a pole, and 3 ro. 25 po. of an acre?

Answers: $\frac{67}{560}$, $\frac{1}{6}$ and $\frac{29}{32}$.

(3) Express 5 bush., 3 pks., 1 gal., as the fraction of a quarter; and 2 wks., 5 days, 18 hrs., as the fraction of a year of 365 days.

Answers: $\frac{47}{64}$ and $\frac{79}{1460}$.

(4) Reduce $\frac{2}{5}$ of 2*s*. $4\frac{1}{2}$*d*. to the fraction of a half crown; and 9*s*. $10\frac{1}{2}$*d*. to the fraction of 13*s*. $2\frac{1}{4}$*d*.

Answers: $\frac{19}{50}$ and $\frac{158}{211}$.

(5) Find the simple fraction of £1. which expresses the sum of $\frac{1}{3}$ of $\frac{7}{10}$ of 13*s*. 4*d*. and $\frac{3}{4}$ of $\frac{5}{7}$ of 10*s*. 6*d*.

Answer: $\frac{629}{1440}$.

(6) Compare the values of $\frac{1}{21}$ of a pound, $\frac{1}{22}$ of a guinea, and $\frac{1}{4}$ of 3*s*. $9\frac{1}{2}$*d*.

Answers: $\frac{7040}{7392}$, $\frac{7056}{7392}$ and $\frac{7007}{7392}$.

(7) Reduce $\dfrac{3\frac{1}{5}}{1\frac{1}{18}}$ of $\left\{\frac{19}{120} \text{ of £1.} - \frac{7}{48} \text{ of 1s.}\right\}$, to the fraction of a moidore.

Answer: $\frac{1885}{5832}$.

94. *If the species of the unit be given, the value of a fraction of it may be expressed by means of its known parts.*

Rule. Multiply the numerator of the fraction by the number of parts of the next inferior denomination

which are equivalent in value to the unit, divide the product by the denominator, and the quotient is the required number of parts of that denomination: proceed in the same way with the remainder, if any, and the parts of the next denomination will be found: and repeat this process till the lowest denomination to which the unit is capable of being reduced, is obtained.

For, if the fraction proposed be $\frac{5}{6}$ of a yard, we have

$$\overset{\text{yds.}}{\frac{5}{6}} = \frac{5}{6} \times \overset{\text{feet.}}{\frac{3}{1}} = \overset{\text{ft.}}{\frac{15}{6}} = \overset{\text{ft.}}{2\tfrac{1}{2}}:$$

$$\overset{\text{ft.}}{\frac{1}{2}} = \frac{1}{2} \times \overset{\text{inches.}}{\frac{12}{1}} = \overset{\text{in.}}{\frac{12}{2}} = \overset{\text{in.}}{6}:$$

and therefore the value of $\frac{5}{6}$ of a yard, expressed in the known parts of a yard, is 2 ft. 6 in., or 30 in.

95. The preceding articles enable us to find the value of the sum or difference of fractional parts of magnitudes, of the same kind.

Ex. Required the sum and difference of $\frac{2}{3}$ of a pound, and $\frac{4}{9}$ of a guinea.

$$\text{Here, } \frac{2}{3} \text{ of a pound} = \frac{2}{3} \text{ of } \overset{s.}{20} = \overset{s.}{\frac{40}{3}} = \overset{s.}{13} \,.\, \overset{d.}{4}:$$

$$\frac{4}{9} \text{ of a guinea} = \frac{4}{9} \text{ of } 21 = \frac{84}{9} = 9 \,.\, 4:$$

$$\text{therefore the sum} = \overset{s.}{13} \,.\, \overset{d.}{4} + \overset{s.}{9} \,.\, \overset{d.}{4} = \overset{£.}{1} \,.\, \overset{s.}{2} \,.\, \overset{d.}{8}:$$

$$\text{and the difference} = 13 \,.\, 4 - 9 \,.\, 4 = 0 \,.\, 4 \,.\, 0.$$

The same results may also be obtained as follows:

$$\text{since } \overset{g.}{\frac{4}{9}} = \frac{4}{9} \times \overset{£.}{\frac{21}{20}} = \overset{£.}{\frac{7}{15}}, \text{ we have}$$

$$\text{the sum} = \frac{2}{3} + \frac{7}{15} = \frac{17}{15} = \overset{£.}{1} \,.\, \overset{s.}{2} \,.\, \overset{d.}{8}:$$

$$\text{the difference} = \frac{2}{3} - \frac{7}{15} = \frac{3}{15} = 0 \,.\, 4 \,.\, 0.$$

Examples for Practice.

(1) Find the values of $\frac{3}{5}$ of a pound, $\frac{5}{9}$ of a shilling, and $\frac{5}{18}$ of a guinea.

Answers: 12*s.*; $6\frac{1}{2}$*d.* $\frac{2}{3}$, and 5*s.* 10*d.*

(2) Required the values of $\frac{5}{7}$ cwt., $\frac{5}{14}$ qrs., and $\frac{3}{8}$ lbs.

Answers: 2 qrs. 24 lbs.; 10 lbs., and 6 oz.

(3) What is the number of feet in $\frac{4}{5}$ of a mile; and the number of yards in $\frac{7}{8}$ of a league?

Answers: 4224 ft., and 4620 yds.

(4) Required the values of $\frac{1}{32}$ qrs., $\frac{3}{8}$ bush. and $\frac{5}{7}$ pks.

Answers: 1 pk.; 1 pk. 1 gal., and 1 gal. 1 qt. $1\frac{3}{7}$ pts.

(5) What are the values of $\frac{2}{15}$ of a month of 28 days, and $\frac{5}{18}$ of a week?

Answers: 3 days. 17 hrs. 36 min., and 1 day. 22 hrs. 40 min.

(6) Find the value of $\dfrac{3}{4}$ of a guinea $+ \dfrac{3}{8}$ of a crown $+ \dfrac{3}{5}$ of 7*s.* 6*d.* $- \dfrac{3}{4}$ of 2*d.*

Answer: £1. 2*s.*

RULES OF PRACTICE.

96. We shall here shew how the primitive fractions, as defined in article (70), may be applied to the *practical* calculation of prices, when the price of an unit of any denomination is supposed to be given: and the tediousness of the enunciations of the rules at length, will be a sufficient excuse for the mere *indication* of the processes employed, by means of examples.

(1) *Simple Practice.*

Ex. 1. Required the value of 1298 at 17*s.* $9\frac{3}{4}$*d.* each, where the unit may be of any denomination whatever.

Here, we shall have no difficulty in tracing the reason of the following process:

s. d.	£.	£. s. d.	£.
10 . 0	$\frac{1}{2}$	1298 . 0 . 0	= price at 1:
5 . 0	$\frac{1}{2}$	649 . 0 . 0	= price at 10*s*.:
2 . 6	$\frac{1}{2}$	324 . 10 . 0	= price at 5*s*.:
0 . 3	$\frac{1}{10}$	162 . 5 . 0	= price at 2*s*. 6*d*.:
0 . 0$\frac{3}{4}$	$\frac{1}{4}$	16 . 4 . 6	= price at 3*d*.:
		4 . 1 . 1$\frac{1}{2}$	= price at $\frac{3}{4}$*d*:
		£ 1156 . 0 . 7$\frac{1}{2}$	= price at 17*s*. 9$\frac{3}{4}$*d*.:

where it is observed that the denomination of the result is the same as that of the unit assumed, which is here £1: and it is generally most convenient, when possible, to use the aliquot parts of the denomination next superior to the highest denomination of the price proposed.

Ex. 2. Find the value of 750 at £5. 8*s*. 4*d*.

Here, proceeding as before, and by Compound Multiplication, we have the following solutions:

By Practice.

s. d.	£.	
6 . 8	$\frac{1}{3}$	750
		5
		3750
1 . 8	$\frac{1}{4}$	250
		62 . 0
		£ 4062 . 10

By Compound Multiplication.

£. s. d.	
5 . 8 . 4	
	$10 \times 5 \times 5 \times 3 = 750$
54 . 3 . 4	
5	
270 . 16 . 8	
5	
1354 . 3 . 4	
3	
£ 4062 . 10 . 0	

and the smaller number of figures employed in the former compared with the latter, proves the *practical* advantage of the method.

(2) *Compound Practice.*

Ex. 1. What is the price of 3cwt. 2qrs. 16lbs. at £3. 7*s*. 8*d*. per cwt?

Here, the following process will be manifest:

		£.	s.	d.	
2 qrs.	$\frac{1}{2}$	3 .	7 .	8	= price of 1 cwt:
				3	
		10 .	3 .	0	= price of 3 cwt:
14 lbs.	$\frac{1}{4}$	1 .	13 .	10	= price of 2 qrs. or $\frac{1}{2}$ of 1 cwt:
2 lbs.	$\frac{1}{7}$	0 .	8 .	$5\frac{1}{2}$	= price of 14 lbs. or $\frac{1}{4}$ of 2 qrs:
		0 .	1 .	$2\frac{1}{2}$	= price of 2 lbs. or $\frac{1}{7}$ of 14 lbs.:
		£ 12 .	6 .	6	= price of 3 cwt. 2 qrs. 16 lbs.

Ex. 2. If a servant's wages be £25. 15*s*. for 12 months, how much will he receive for 7 months?

Proceeding as before, we have

		£.	s.	d.	
6 mo.	$\frac{1}{2}$	25 .	15 .	0	= wages for 12 months:
1 mo.	$\frac{1}{6}$	12 .	17 .	6	= wages for 6 months:
		2 .	2 .	11	= wages for 1 month:
		£ 15 .	0 .	5	= wages for 7 months:

and here, as well as in the preceding examples, the operations may be divested of the explanations affixed to their right hand, without much affecting the clearness of the principles.

Ex. 3. Required the value of $2937\frac{1}{2}$ at $10\frac{3}{4}d$.

d.	s.	s.
6	$\frac{1}{2}$	2937
4	$\frac{1}{3}$	1468 . 6
$\frac{1}{2}$	$\frac{1}{8}$	979 . 0
$\frac{1}{4}$	$\frac{1}{2}$	122 . $4\frac{1}{2}$
		61 . $2\frac{1}{4}$
$\frac{1}{2}$ of $10\frac{3}{4}$ =		0 . $5\frac{1}{4}$, $\frac{1}{2}f.$
	2,0)	263,1 . 6, $\frac{1}{2}f.$
		£ 131 . 11 . 6, $\frac{1}{2}f.$

where the former part of the operation is simple practice, and the latter compound.

Examples for Practice.

(1)	2710 at $1\frac{1}{2}d.$	Ans.:	£16. 18*s*. 9*d*.
(2)	3467 at $3\frac{3}{4}d.$	Ans.:	£54. 3*s*. $5\frac{1}{4}d.$
(3)	659 at 1*s*. $7\frac{3}{4}d.$	Ans.:	£54. 4*s*. $7\frac{1}{4}d.$
(4)	328 at 8*s*. $5\frac{1}{2}d.$	Ans.:	£138. 14*s*. 4*d*.
(5)	7351 at 14*s*. $9\frac{1}{4}d.$	Ans.:	£5429. 0*s*. $4\frac{3}{4}d.$
(6)	537 at £1. 7*s*. $2\frac{1}{2}d.$	Ans.:	£730. 10*s*. $10\frac{1}{2}d.$
(7)	2937 at £2. 11*s*. $10\frac{3}{4}d.$	Ans.:	£7620. 18*s*. $0\frac{3}{4}d.$
(8)	$7432\frac{1}{2}$ at 13*s*. $6\frac{1}{2}d.$	Ans.:	£5032. 8*s*. $5\frac{1}{4}d.$
(9)	$1530\frac{1}{4}$ at 15*s*. 9*d*.	Ans.:	£1205. 1*s*. $5\frac{1}{4}d.$
(10)	$6147\frac{3}{4}$ at 17*s*. $6\frac{1}{2}d.$	Ans.:	£5392. 1. $9\frac{1}{4}d, \frac{1}{2}.$
(11)	$217\frac{1}{4}$ at £2. 17*s*. $7\frac{1}{2}d.$	Ans.:	£625. 19*s*. $0\frac{1}{4}d, \frac{1}{2}.$
(12)	$769\frac{3}{5}$ at £1. 12*s*. 6*d*.	Ans.:	£1250. 12*s*. 0*d*.
(13)	$674\frac{5}{8}$ at £3. 19*s*. $6\frac{1}{2}d.$	Ans.:	£2683. 0*s*. $9\frac{1}{2}d, \frac{1}{4}.$

(14) Find the price of 2 cwt., 3 qrs., 12 lbs., at £1. 7*s*. 6*d*. a cwt.

Answer: £3. 18*s*. $6\frac{3}{4}d, \frac{3}{7}.$

(15) Find the cost of 57 cwt., 3 qrs., 14 lbs., at £5. 9*s*. 6*d*. a cwt.

Answer: £316. 17*s*. $3\frac{3}{4}d.$

(16) What is the value of 45 oz., 6 dwt., 7 grs., at 5*s*. 10*d*. an ounce?

Answer: £13. 4*s*. $4\frac{1}{48}d.$

(17) Required the value of 16 yds., 2 ft., 10 in., at 2*s*. $6\frac{1}{2}d.$ a yard.

Answer: £2. 3*s*. $0\frac{3}{4}d, \frac{2}{9}.$

(18) Find the value of 44 ac., 2 ro., 25 po., at £55. 16*s*. $7\frac{1}{2}d.$ an acre.

Answer: £2493. 4*s*. $3\frac{1}{4}d, \frac{11}{16}.$

MISCELLANEOUS QUESTIONS.

(1) If I have an eighth of a fifth part of £2000., what is the value of my share?

$$\text{Here, } \frac{1}{8} \text{ of } \frac{1}{5} = \frac{1 \times 1}{8 \times 5} = \frac{1}{40}:$$

therefore the value of my share is

$$\frac{1}{40} \text{ of } \overset{£.}{2000} = \frac{\overset{£.}{2000}}{40} = \overset{£.}{50}:$$

or, taking it in another point of view,

$$\text{we have } \frac{1}{5} \text{ of } \overset{£.}{2000} = \frac{\overset{£.}{2000}}{5} = \overset{£.}{400}:$$

$$\text{whence } \frac{1}{8} \text{ of } 400 = \frac{400}{8} = 50, \text{ as before.}$$

(2) The aggregate of $\frac{2}{3}$ and $\frac{3}{5}$ of a sum of money is £133: what is the sum?

$$\text{Here, } \frac{2}{3} + \frac{3}{5} = \frac{10+9}{15} = \frac{19}{15}:$$

$$\text{therefore, } \frac{19}{15} \text{ of the sum is £133:}$$

$$\text{whence, } \frac{1}{15} \text{ of the sum is } \frac{1}{19} \text{ of £133} = £7:$$

and the sum itself $= 7 \times 15 =$ £105, as may be easily verified.

(3) Find the fraction which, when multiplied by $\frac{2}{3}$ of $\frac{4}{5}$ of $3\frac{1}{2}$, gives a result equal to $\frac{7}{9}$.

$$\text{First, } \frac{2}{3} \text{ of } \frac{4}{5} \text{ of } 3\tfrac{1}{2} = \frac{2}{3} \times \frac{4}{5} \times \frac{7}{2} = \frac{28}{15}:$$

$$\text{therefore, the required fraction} \times \frac{28}{15} = \frac{7}{9}:$$

but since, when equal quantities are multiplied by the same quantity, the results are evidently equal, we have

$$\text{the required fraction} \times \frac{28}{15} \times \frac{15}{28} = \frac{7}{9} \times \frac{15}{28} = \frac{5}{12}:$$

$$\text{that is, the required fraction} = \frac{5}{12}, \text{ because } \frac{28}{15} \times \frac{15}{28} = 1.$$

(4) Find what number of times £24. 16*s*. $4\frac{1}{2}$*d*. is contained in £335. 1*s*. $0\frac{3}{4}$*d*.

$$\text{Here,}\quad \overset{\pounds.}{24} \,.\, \overset{s.}{16} \,.\, \overset{d.}{4\tfrac{1}{2}} = \frac{\overset{\pounds}{23826}}{960}:$$

$$\text{and } 335 \,.\, 1 \,.\, 0\tfrac{3}{4} = \frac{321651}{960}:$$

whence the required number of times

$$= \frac{\overset{\pounds}{321651}}{960} \div \frac{\overset{\pounds}{23826}}{960} = \frac{321651}{960} \times \frac{960}{23826}$$

$$= \frac{321651}{23826} = \frac{27}{2} = 13\tfrac{1}{2};$$

that is, £335. 1*s*. $0\frac{3}{4}$*d*. is equal to $13\frac{1}{2}$ times £24. 16*s*. $4\frac{1}{2}$*d*.: and this being regarded as a compound *unit* and represented by 1, the former will be represented by $13\frac{1}{2}$.

(5) A person possessed of $\frac{2}{5}$ths of a coal mine, sells $\frac{3}{4}$ths of his share for £2000; what is the whole mine worth?

Here, if the mine be considered the unit and be represented by 1,

$$\text{we have } \frac{3}{4} \text{ of } \frac{2}{5} \text{ of it} = \frac{3}{10},$$

the fraction of it sold for £2000: that is $\frac{3}{10}$ is worth £2000:

therefore $\frac{1}{10}$ is worth $\frac{1}{3}$ of £2000, or £666. 13*s*. 4*d*.:

and 1, or the whole mine, is worth

(£666. 13*s*. 4*d*.) × 10 = £6666. 13*s*. 4*d*.

(6) *A* can do a piece of work in 5 days, *B* in 6 and *C* in 7: how much of it can they jointly do in 2 days?

Assuming the piece of work to be represented by the unit or 1, we have

$\frac{1}{5}$ = part done by *A* in 1 day,

$\frac{1}{6}$ = ... *B* ....

$\frac{1}{7}$ = ... *C* ....

$$\text{therefore } \frac{1}{5} + \frac{1}{6} + \frac{1}{7} = \frac{42 + 35 + 30}{210} = \frac{107}{210}$$

is the part done jointly by A, B and C in 1 day: whence the work done by them jointly in 2 days, will be

$$\frac{107}{210} \times 2 = \frac{214}{210} = 1\frac{2}{105};$$

that is, they could finish the whole work in 2 days, and $\frac{2}{105}$ of the same work besides.

Hence also, the time in which they would exactly complete the work is

$$1 \div \frac{107}{210} = 1 \times \frac{210}{107} = \frac{210}{107} = 1\frac{103}{107} \text{ days.}$$

(7) One half of the trees in an orchard are apple trees, one fourth are pear trees, one sixth plum trees, and there are 50 cherry trees: what number of trees does it contain?

Representing the number of trees in the orchard by the unit or 1, we have

$\frac{1}{2}$ = number of apple trees:
$\frac{1}{4}$ = number of pear trees:
$\frac{1}{6}$ = number of plum trees:

and the sum of these numbers $= \frac{1}{2} + \frac{1}{4} + \frac{1}{6} = \frac{11}{12}$:

whence the number of cherry trees $= 1 - \frac{11}{12} = \frac{1}{12}$:

that is, $\frac{1}{12}$ of the whole number of trees = 50; and the whole number is therefore $= 50 \times 12 = 600$.

This is easily verified, for,

the number of apple trees $= \frac{1}{2}$ of $600 = 300$:
the number of pear trees $= \frac{1}{4}$ of $600 = 150$:
the number of plum treees $= \frac{1}{6}$ of $600 = 100$:
the number of cherry trees = ... = 50:

and the number of trees in the orchard = 600.

Examples for Practice.

(1) If $\frac{3}{16}$ of a lottery ticket cost £4. 10*s*., what is the price of $\frac{1}{5}$ of a ticket?

Answer: £4. 16*s*.

(2) The owner of $\frac{4}{17}$ of a ship, sold $\frac{3}{11}$ of $\frac{2}{9}$ of his share for £$12\frac{4}{33}$; what would $\frac{2\frac{1}{2}}{4\frac{1}{4}}$ of $\frac{2}{5}$ of it cost, at the same rate?

Answer: £200.

(3) Express a degree of $69\frac{1}{2}$ miles in metres, where 32 metres are equal to 35 yards.

Answer: $111835\frac{3}{7}$ metres.

(4) If I import 5763 bushels of wheat for £1800. 18*s*. 9*d*., and pay an import duty of $10\frac{1}{2}$ per cent. on the money expended, what is the duty per bushel?

Answer: $7\frac{3}{4}$*d*.

(5) Find the value of the metre of France, in terms of the foot of Cremona, if 48 Cremonese feet = 56 English feet, and if the metre be $39\frac{371}{1000}$ English inches.

Answer: $2\frac{11871}{14000}$ feet.

(6) What number is that, whereof the part expressed by $\frac{1}{3}+\frac{1}{4}+\frac{1}{6}$ is 45?

Answer: 60.

(7) A post has one-fourth of its length in the mud, one-third in the water, and 10 feet above the water: find its whole length.

Answer: 24 feet.

(8) *A* met two beggars *B* and *C*, and having $\frac{3\frac{7}{11}}{4\frac{2}{7}}$ of $\frac{10\frac{5}{7}}{7\frac{1}{2}}$ of $\frac{77}{540}$ of a moidore in his pocket, gave $\frac{1}{7}$ of $\frac{3}{4}$ of it to *B*, and $\frac{3}{5}$ of the remainder to *C*: what did each receive?

Answer: *B* received 6*d*., and *C* had 2*s*. 6*d*.

(9) *A* had at first £1 . 8*s*.; and *B*, when he had paid $2\frac{3\frac{1}{7}}{1\frac{3}{5}}$ of £1. 11*s*. 6*d*. to *A*, found that he had remaining $\frac{1}{43}$ of what *A* then had: what had *B* at first?

Answer: £7. 8*s*.

(10) If a cask be emptied by two taps in 4 and 6 hours respectively, in what time will it be emptied by both of them together, the rates of efflux remaining the same throughout?

Answer: 2 hrs. 24 min.

(11) *A*, *B* and *C* can perform a piece of work in 12 hours: also *A* and *B* can do it in 16 hours, and *A* and *C* in 18 hours: what part of the work can *B* and *C* do in $9\frac{1}{7}$ hours?

Answer: $\frac{4}{9}$.

(12) Ten excavators can dig 12 loads of earth in 16 hours, whilst 12 others can dig only 9 loads in 15 hours: find in what time they will jointly dig 100 loads.

Answer: $74\frac{2}{27}$ hours.

(13) A cistern is filled by two spouts in 20 and 24 minutes respectively, and emptied by a tap in 30 minutes: what portion of it will be filled in 15 minutes, when they are all left open together, the influx and efflux being uniform?

Answer: $\frac{7}{8}$.

(14) In an orchard, $\frac{1}{3}$ of the trees are apple trees, $\frac{1}{4}$ pear trees, $\frac{1}{5}$ cherry trees, $\frac{1}{6}$ filbert trees, and there are 12 walnut trees: what is the number of each sort?

Answer: 80 apple trees, 60 pear trees, 48 cherry trees, 40 filbert trees, and 12 walnut trees.

(15) A person after paying away one-third of his money together with £10., finds that he has remaining £15. more than its half: what money had he?

Answer: £150.

(16) A farmer pays a corn-rent of 5 quarters of wheat and 3 quarters of barley, Winchester measure: what is his rent, wheat being at 60*s*., and barley at 54*s*. per quarter, imperial measure, it being assumed that 32 imperial gallons are equivalent to 33 Winchester gallons?

Answer: £22. 8*s*.

CHAPTER V.

THE THEORY OF DECIMALS,

COMMONLY CALLED DECIMAL FRACTIONS.

97. Def. In the articles upon the Notation of Integers, it has been seen that the figures in the units' place alone retain their *absolute* values, whilst the *local* values of figures in other situations increase *tenfold* for every individual figure we advance towards the left hand from that place. Hence, therefore, in beginning at the *left* figure of any number and proceeding towards the *right* hand, it necessarily follows that the *local* value of every successive figure will be a *tenth* part of that which immediately precedes it: and if we suppose figures to be situated to the right of the units' place, and this kind of tenfold *subdivision* to be extended to them, it is manifest that the local values of such figures in order from the place of units' will be a *tenth*, a *hundredth*, a *thousandth*, &c., parts of their absolute values.

This consideration will therefore enable us to represent integers and fractions by one uniform system of notation, by merely fixing upon the place of units: and whilst *Integers* are expressed by figures in the units' place and in places to the *left* of it, *Fractions* will be represented by figures situated in places on the *right* of the units, called the places of *tenths, hundredths, thousandths*, &c.

In this manner originates the System of *Decimal Notation*, being merely an extension of the Notation of Integers: and from the circumstance of its representing only *tenths, hundredths, thousandths*, &c., of the unit, all fractions belonging to it are termed *Decimals*, or, *Decimal Fractions*, in contradistinction to *Vulgar Fractions*, whereof the denominations may be any parts we please. Whence *Decimals* are sometimes *defined* to be *Fractions* whose denominators are 10, 100, 1000, &c.

NOTATION, &c. OF DECIMALS.

98. The preceding definition implying the necessity of *fixing* the units' place, if we place 1 in that situation, the following Scheme analogous to the Numeration Table, will point out the denominations of the figures to the left and right of it; and it may manifestly be extended so as to include both integers and fractions of all local values whatever:

&c.	Thousands.	Hundreds.	Tens.	Units.	Tenths.	Hundredths.	Thousandths.	&c.
&c.	4	3	2	1	2	3	4	&c.

and a mixed quantity, thus formed of integers and fractions, is, in practice, separated into its *integral* and *fractional* portions by means of a *full point* placed on the right of the units' place, which dispenses with the description of the local denominations above given.

Thus, in 4321.2345, the figures 4321, on the *left* of the point, denote so many integers; and the figures 2345 on the *right* of it, so many fractions, namely, 2 *tenths*, 3 *hundredths*, 4 *thousandths*, and so on: and the *expressing* and *reading* of Decimal Fractions will evidently be conducted upon the respective principles of the *Notation* and *Numeration* of Integers: also, inasmuch as integers denote assemblages of two or more *units*, these decimals represent according to a similar law, assemblages of two or more tenth, hundredth, &c., *parts* of an unit.

Relation of Decimal Fractions to Vulgar Fractions.

99. From the statements made in the preceding articles, it is obvious that every magnitude made up of one or more decimals is equivalent to, and may be expressed by, one or more vulgar fractions having 10, 100, 1000, &c., for their denominators: and that all *mixed* quantities expressed decimally may be represented by means of *whole* numbers and *vulgar fractions* of similar denominations.

100. *Every decimal fraction may be expressed exactly by a vulgar fraction: and every mixed decimal fraction by a mixed vulgar fraction.*

For, from what has just been said, we have

$$.327 = \frac{3}{10} + \frac{2}{100} + \frac{7}{1000} = \frac{327}{1000}:$$

$$.0459 = \frac{0}{10} + \frac{4}{100} + \frac{5}{1000} + \frac{9}{10000} = \frac{459}{10000}:$$

$$13.816 = 13 + \frac{8}{10} + \frac{1}{100} + \frac{6}{1000} = \frac{13816}{1000}:$$

and hence we infer that a decimal will always be equivalent to the vulgar fraction formed by taking it, considered *integral*, for the numerator, and having 1, with as *many* ciphers as there are *decimal places* in it, for the denominator. In these instances, we see that the reduction to a common denominator, so tedious in vulgar fractions, may be entirely dispensed with, and the *immediate* comparison of fractional quantities is one of the great advantages of the system.

Since $.327 = \frac{327}{1000}$, it is evident that .327 may be read as 327 *thousandths*, the fraction having always the denomination of its *last* figure on the right hand: and conversely, every vulgar fraction having 10, 100, 1000, &c., for its denominator, may be immediately represented by a decimal fraction, by beginning at the figure on the right hand of the numerator, and pointing off as many decimal places, supplied with ciphers towards the *left*, if necessary, as there are ciphers in the denominator.

101. *Ciphers annexed to the right hand of a decimal fraction have no effect upon its value.*

Thus, $.37 = \frac{37}{100}$, $.370 = \frac{370}{1000} = \frac{37}{100}$,

$$.3700 = \frac{3700}{10000} = \frac{37}{100}, \text{ and so on:}$$

as appears also from the consideration, that there are *no* thousandths, &c., in addition to the tenths and hundredths expressed by .37.

102. *Every cipher affixed to the left hand of a decimal fraction diminishes its value tenfold.*

Thus, $.43 = \frac{43}{100}$, $.043 = \frac{43}{1000}$, $.0043 = \frac{43}{10000}$, &c.;

where each succeeding fraction is a tenth part of that which immediately precedes it: and indeed this is also evident from the circumstance of every figure being reduced one denomination lower by means of each cipher.

Hence also, *Multiplication* and *Division* by 10, 100, 1000, &c., are immediately effected, by shifting the decimal point 1, 2, 3, &c., places towards the *right* and *left* respectively.

103. *Every vulgar fraction may be expressed either accurately or approximately by a decimal.*

Let us take the fractional quantities $\frac{3}{8}$ and $4\frac{7}{125}$: then, by reduction of vulgar fractions, we have

$$\frac{3}{8} = \frac{3000}{8000} = \frac{\frac{1}{8}(3000)}{1000} = \frac{375}{1000} = .375:$$

$$\text{and } 4\frac{7}{125} = 4 + \frac{7000}{125000} = 4 + \frac{\frac{1}{125}(7000)}{1000}$$

$$= 4 + \frac{56}{1000} = 4 + .056 = 4.056:$$

whence we have the following rule.

RULE. Divide the numerator of the fraction with as many ciphers annexed to the right of it, as may be deemed necessary, by the denominator: and the quotient comprising as many decimal places as there are ciphers annexed, will be the decimal required.

If the prescribed division do not terminate, neither is the corresponding decimal *finite*, and the vulgar fraction is expressed only *approximately* by the decimal fraction thus found: three or four ciphers are generally sufficient for all practical purposes, but the approximation will be nearer, the further the division is continued, inasmuch as, by every succeeding step of the operation, a decimal fraction of an inferior denomination is added to the value already obtained.

Examples for Practice.

(1) Express .3, .073, .0059 and 21.70947 in the form of vulgar fractions.

$$\text{Answers: } \frac{3}{10}, \frac{73}{1000}, \frac{59}{10000} \text{ and } \frac{2170947}{100000}.$$

(2) Transform the decimals, .5, .75, .625 and .1875, to vulgar fractions in their lowest terms.

Answers: $\frac{1}{2}$, $\frac{3}{4}$, $\frac{5}{8}$ and $\frac{3}{16}$.

(3) Find the simplest vulgar fractions equivalent to the decimals: .00625, .1015625 and .0109375.

Answers: $\frac{1}{160}$, $\frac{13}{128}$ and $\frac{7}{640}$.

(4) What vulgar fractions are equivalent to the mixed decimals:

1.075, 3.01875, 7.0046875 and 13.0005859375?

Answers: $\frac{43}{40}$, $\frac{483}{160}$, $\frac{4483}{640}$ and $\frac{66563}{5120}$.

(5) Convert the vulgar fractions,

$\frac{5}{8}$, $\frac{15}{128}$, $\frac{13}{1600}$ and $\frac{4}{625}$ to decimals.

Answers: .625, .1171875, .008125 and .0064.

(6) What are the decimal fractions equivalent to $\frac{4}{25}$, $\frac{9}{125}$ and $\frac{17}{2560}$?

Answers: .16, .072 and .006640625.

(7) Represent the approximate values of

$\frac{1}{3}$, $\frac{2}{7}$, $\frac{3}{11}$ and $\frac{5}{13}$,

to three or more places of decimals.

Answers: .333 &c., .285 &c., .272 &c., and .384615 &c.

I. ADDITION OF DECIMALS.

Rule. Place the quantities so that all the decimal points may be in the same vertical line, to insure the combination of those of the same denominations: and add them together as in integers, taking care to place the decimal point in the sum, immediately under those of the quantities proposed.

Examples.

$\begin{array}{r}.419\\.0256\\.08\\.21734\\\hline .74194\end{array}$	$\begin{array}{r}25.6\\4.805\\.009\\653.27\\\hline 683.684\end{array}$

Proof by Vulgar Fractions.

Using only the latter example, we have

$$25.6 = \frac{256}{10} = \frac{25600}{1000}:$$

$$4.805 = \frac{4805}{1000}:$$

$$.009 = \frac{9}{1000}:$$

$$653.27 = \frac{65327}{100} = \frac{653270}{1000}:$$

whence the sum

$$= \frac{25600 + 4805 + 9 + 653270}{1000} = \frac{683684}{1000} = 683.684,$$

as above.

Hence decimals are sometimes said to be reduced to a *common denominator*, when ciphers are supplied so that there is the *same* number of decimal places in each.

II. SUBTRACTION OF DECIMALS.

Rule. Place the less quantity under the greater as in Addition; suppose the ciphers to be supplied, if necessary, in the upper line; and the difference, found as in integers, will have as many decimal places as are contained in each, either expressed or understood.

Examples.

$\begin{array}{r}.7053\\.6729\\\hline .0324\end{array}$	$\begin{array}{r}41.62\\34.917\\\hline 6.703\end{array}$

Proof by Vulgar Fractions.

In the latter of these examples we have

$$41.62 - 34.917 = \frac{4162}{100} - \frac{34917}{1000}$$

$$= \frac{41620 - 34917}{1000} = \frac{6703}{1000} = 6.703,$$

as before; and the necessity of supposing the cipher to be supplied is here shewn.

III. MULTIPLICATION OF DECIMALS.

Rule. Multiply together the quantities proposed as if they were integers: and the product will contain as many places of decimals, as there are decimal places in the multiplicand and multiplier together.

Examples.

$$\begin{array}{r} .45 \\ .21 \\ \hline 45 \\ 90 \\ \hline .0945 \end{array} \qquad\qquad \begin{array}{r} 6.27 \\ 15.9 \\ \hline 5643 \\ 3135 \\ 627 \\ \hline 99.693 \end{array}$$

where the former product, found as *whole numbers* would manifestly be *ten thousand* times too great, because 45 and 21 are a *hundred* times as great as .45 and .21 respectively; and therefore the true product is obtained by placing the decimal point *four* places towards the *left* hand, by article (102).

Proof by Vulgar Fractions.

The latter product of the last examples is

$$6.27 \times 15.9 = \frac{627}{100} \times \frac{159}{10} = \frac{99693}{1000} = 99.693,$$

there being always as many ciphers in the denominator of the product, as there are in those of both the factors together.

IV. DIVISION OF DECIMALS.

RULE. Supply the dividend with ciphers to the right hand, if necessary, and divide exactly as in integers: then the quotient will have a number of decimal places equal to the excess of the number of such places in the dividend above that in the divisor.

Examples.

$$.012)\,.241728 \quad \underline{20.144}$$

$$\begin{array}{r} 2.5)\,.1875(\,.075 \\ \underline{175} \\ 125 \\ \underline{125} \end{array}$$

wherein the number of decimal places in the quotient is the excess of the number of decimal places in the dividend above that in the divisor, because the divisor and quotient must together comprise as many as the dividend, by the last rule.

Proof by Vulgar Fractions.

Here, $.1875 \div 2.5 = \frac{1875}{10000} \div \frac{25}{10} = \frac{1875}{10000} \times \frac{10}{25} = \frac{75}{1000}$ $= .075$, as before.

If the divisor and dividend have the *same* number of decimal places, the quotient will evidently be an *integer*, as there is no excess: but if there be *more* places in the divisor than in the dividend, ciphers must be supplied so as to render the number in the dividend *not less* than that in the divisor, before the rule can be applied: and the reason of this will be seen in the following example:

$62.5 \div .025 = \frac{625}{10} \div \frac{25}{1000} = \frac{625}{10} \times \frac{1000}{25} = \frac{625}{25} \times \frac{1000}{10} = 25$ $\times 100 = 2500$: where there is annexed to the right of the quotient, obtained as in integers, a number of ciphers equal to the excess of the number of decimal places in the divisor above that in the dividend, the correct quotient being the integral quantity 2500.

If the division do not terminate, *three* or *four* decimal places in the quotient are generally sufficient.

Examples for Practice.

(1) Find the sum of .295, 3.086, 12.87, .0051 and 729.54.

Answer: 745.7961.

(2) Add together 36.053, .0079, .000952, 417, 85.5803 and .0000501.

Answer: 538.6422021.

(3) Find the difference of 27.903 and .054: also, of 7295.06 and 254.738.

Answers: 27.849 and 7040.322.

subtraction

(4) Required the excess of 2.057 above 1.0097, and of 3.025 above .003025.

Answers: 1.0473 and 3.021975.

(5) Required the respective products of .718 and .57: of 16.8 and .0024: and of 144 and .0625.

Answers: .40926, .04032 and 9.

(6) Multiply 270.56 by .37025, and .00579 by 3796.8.

Answers: 100.17484 and 21.983472.

(7) Find the continued product of .275, 2.75 and 27.5.

Answer: 20.796875.

(8) Required the respective quotients of .278831 by .653: of 11.444495 by 4.735, and of .020872522 by .08635.

Answers: .427, 2.417 and .24172.

(9) Divide .0257 by .0041: 325.46 by .0187, and .0719 by 27.53, to three or more places of decimals.

Answers: 6.268 &c., 17404.278 &c. and .00261 &c.

(10) Find the quotients of 1.68 by .024: of 971. 7 by .123, and of 142.025 by .0437; and prove the results by vulgar fractions.

Answers: 70, 7900 and 3250.

REDUCTION OF DECIMALS.

104. A general view having now been taken of decimals, we will next shew how they may be made to change their denominations, when they are considered as belonging to a particular unit; and how they may be adapted to the practical computations, in which they are most frequently employed.

105. *A decimal may always be changed into another, whose denomination shall have a given relation to its own.*

RULE. Multiply or divide the given decimal, by the numbers which connect the various denominations in order, according as the denomination of the required decimal is lower or higher than its own.

For, from what has been said in the reduction of compound quantities, it is evident that

$$\overset{\text{cwt.}}{.16} = \overset{\text{qrs.}}{.16} \times 4 = \overset{\text{qrs.}}{.64}, \text{ and } \overset{\text{lbs.}}{14} = \overset{\text{qrs.}}{\frac{14}{28}} = \overset{\text{qrs.}}{.5} = \overset{\text{cwt.}}{\frac{.5}{4}} = \overset{\text{cwt.}}{.125}.$$

106. *A compound quantity may be exhibited in the form of a decimal whose denomination is given.*

RULE. Divide the lowest denomination by the number which connects it with the next, and to the left of the quotient affix the number of the lowest denomination: and continue this process till the required denomination is obtained.

Let us take 7fur. 25po., and express it as the decimal of a mile: then

$$\overset{\text{po.}}{25} = \overset{\text{fur.}}{\frac{25}{40}} = \overset{\text{fur.}}{.625} = \overset{\text{mi.}}{\frac{.625}{8}} = \overset{\text{mi.}}{.078125}, \text{ and } \overset{\text{fur.}}{7} = \overset{\text{mi.}}{\frac{7}{8}} = \overset{\text{mi.}}{.875}:$$

whence the decimal will be .078125 + .875 = .953125 of a mile: also, the same processes are comprised in the following more convenient and *practical form:*

$$\begin{array}{r} 40)\underline{25.000} \\ 8)\underline{7.625000} \\ \underline{.953125} \end{array}$$

which suggests the rule.

107. *The value of a decimal fraction may be expressed by means of the known parts of its unit.*

RULE. Multiply the proposed decimal by the numbers which connect the successive denominations in order; and the integral parts of the products *taken out*, as they occur, will be the value required.

For, to find the value of .655 of a day, we have

$$\overset{\text{days.}}{.655} = \overset{\text{hrs.}}{.655 \times 24} = \overset{\text{hrs.}}{15.72}:$$

$$\overset{\text{hrs.}}{.72} = \overset{\text{min.}}{.72 \times 60} = \overset{\text{min.}}{43.2}:$$

$$\overset{\text{min.}}{.2} = \overset{\text{sec.}}{.2 \times 60} = \overset{\text{sec.}}{12}:$$

that is, 15 hrs. 43 min. 12 sec., is the value required: and the following *form* amounts to the same thing:

$$\begin{array}{rr}
 & \text{days.} \\
 & .655 \\
 & 24 \\
\hline
 & 2620 \\
 & 1310 \\
\hline
\text{hrs.} & 15.720 \\
 & 60 \\
\hline
\text{min.} & 43.200 \\
 & 60 \\
\hline
\text{sec.} & 12.000 \\
\hline
\end{array}$$

Examples for Practice.

(1) Express £.00375 as the decimals of a shilling and a penny.

Answers: .075*s*., and .9*d*.

(2) What decimals of a pound are 8.4 of a penny; and .4068 of a farthing?

Answers: £.035, and £.00042375.

(3) Reduce 2.15 lbs. to the decimal of 1 cwt., and 24 yards to the decimal of a mile.

Answers: .01919 &c., and .0136 &c.

(4) Reduce 7 oz. 4 dwts. to the decimal of 1 lb., and 2 qrs. $3\frac{1}{3}$ nls. to the decimal of an English ell of five quarters.

Answers: .6, and .555 &c.

(5) Reduce 12 hrs. 55 min. $23\frac{1}{13}$ sec. to the decimal of a day, and 5 days. 12 hrs. 25 min. 37.92 sec. to the decimal of a week.

Answers: .538461 &c., and . 788257 &c.

(6) Express 12*s*. $6\frac{3}{4}$*d*., 15*s*. $9\frac{3}{4}$*d*. and £4. 13*s*. $4\frac{1}{2}$*d*. as decimals of £1.

Answers: .628125, .790625, and 4.66875.

(7) Reduce 1.1*s*. to the decimal of 10*s*., and 5*s*. to to the decimal of 13*s*. 4*d*.

Answers: .11 and .375.

(8) Find the values of .45 of £1., .16875 of £1. and 2.36875 of £1.

Answers: 9*s*., 3*s*. $4\frac{1}{2}$*d*., and £2. 7*s*. $4\frac{1}{2}$*d*.

(9) Required the values of £ .5675, .375 cwt., .6875 yds. and 13.3375 acres.

Answers: 11*s*. $4\frac{1}{5}$*d*., 1 qr. 14 lbs., 2 qrs. 3 na., and 13 ac. 1 ro. 14 po.

(10) What are the values of .203125 qrs., and .73625 bush.?

Answers: 1 bush. 2 pks. 1 gal., and 2 pks. 1 gal. $3\frac{14}{25}$ qts.

(11) What are the values of .07 of £2. 10*s*., and of .0474609375 of £10. 13*s*. 4*d*.?

Answers: 3*s*. 6*d*., and 10*s*. $1\frac{1}{2}$*d*.

(12) Find the value of .5 shillings + .7 crowns + .125 pounds.

Answer: 6*s*. 6*d*.

(13) Reduce £24. 16*s*. $4\frac{1}{2}$*d*. and £167. 10*s*. $6\frac{1}{4}$*d*. $\frac{1}{2}$, to decimals of the same denomination; and find how often the former is contained in the latter.

Answer: 6.75.

(14) Express .375 of a guinea + $\frac{3}{16}$ of a crown + .3 of 7*s*. 6*d*. $-\frac{3}{8}$ of 2*d*. as the decimal of 16*s*.

Answer: .6875.

RECURRING DECIMALS.

108. Def. In the conversion of a vulgar fraction into a decimal, if the division performed according to the rule laid down in article (103), do not terminate, but the figures of the quotient continually recur in some certain order, the result is called a *recurring* or *circulating* decimal: the quantity repeated is styled its *period*, and is frequently termed a *simple* or *compound repetend*, according as it consists of *one* or *more* figures: and the *extent* of the period is denoted by means of single points or dots placed over the *first* and *last* of the figures which compose it. If the quotient comprise other figures besides those which are repeated, it is called a *mixed* circulating decimal, consisting of a *non-recurring* and a *recurring* part.

Ex. 1. Convert $\frac{1}{3}$ and $\frac{4}{27}$ into decimals.

Proceeding according to the rule, we have

$$3\,)\,\underline{1\,.\,0\,0\,0\,0\ \&c.}\qquad\qquad 27\begin{cases}3\,)\,\underline{4\,.\,0\,0\,0\,0\,0\,0\ \&c.}\\ 9\,)\,\underline{1\,.\,3\,3\,3\,3\,3\,3\ \&c.}\end{cases}$$
$$\quad .\,3\,3\,3\,3\ \&c.\qquad\qquad\qquad .\,1\,4\,8\,1\,4\,8\ \&c.$$

whence, $\frac{1}{3} = .3333$ &c., and $\frac{4}{27} = .148148$ &c.:

the former having the *simple* repetend 3, and the latter the *compound* repetend 148, which being denoted by $\dot{3}$ and $\dot{1}4\dot{8}$ respectively,

$$\text{give } \frac{1}{3} = .\dot{3}, \text{ and } \frac{4}{27} = .\dot{1}4\dot{8}:$$

and these are sometimes termed *pure circulates.*

Ex. 2. What is the decimal corresponding to $\frac{5}{36}$? answer .13

As in the preceding instances, we have

$$36\begin{cases}6\,)\,\underline{5\,.\,0\,0\,0\,0\,0\ \&c.}\\ 6\,)\,\underline{.\,8\,3\,3\,3\,3\ \&c.}\end{cases}$$
$$.\,1\,3\,8\,8\,8\ \&c.$$

whence $\frac{5}{36}$ is equivalent to the *mixed* circulating decimal .13888 &c., the *non-recurring* part being 13 and the *recurring* part 8, and the result is written $\frac{5}{36} = .13\dot{8}$.

Conversely, every pure or mixed circulating decimal must be equal to, and expressible by, a vulgar fraction.

109. *To find the vulgar fraction which shall be equivalent to a pure recurring decimal.*

Let the circulates be .666 &c., and .9696 &c., or $.\dot{6}$ and $.\dot{9}\dot{6}$: then if, for the sake of conciseness, we suppose the symbols x and y to represent their values, we shall have the following operations:

$$\begin{array}{rl|rl} x = & .666 \text{ \&c.} & y = & .9696 \text{ \&c.} \\ 10x = & 6.666 \text{ \&c.} & 100y = & 96.9696 \text{ \&c.} \end{array}$$

whence subtracting in each case, the former from the latter, we obtain

$$\begin{array}{rl|rl} 9x = & 6, & 99y = & 96, \\ \text{and } x = & \frac{6}{9} = \frac{2}{3}: & \text{and } y = & \frac{96}{99} = \frac{32}{33}: \end{array}$$

that is, the vulgar fractions are $\frac{2}{3}$ and $\frac{32}{33}$.

These results may be easily verified, and from them we derive the following rule.

RULE. Make the repetend the *numerator* of a fraction whose *denominator* shall consist of as many *nines* as there are figures in the said repetend, and this reduced to its simplest terms will be the vulgar fraction required.

110. *To find the vulgar fraction which shall represent the value of a mixed recurring decimal.*

Ex. To ascertain the vulgar fractions equivalent to $.2\dot{7}$ and $.24\dot{5}\dot{7}$, we have

$$\begin{array}{rl|rl} x = & .2\dot{7} & y = & .24\dot{5}\dot{7} \\ 10x = & 2.\dot{7} & 100y = & 24.\dot{5}\dot{7} \\ 100x = & 27.\dot{7} & 10000y = & 2457.\dot{5}\dot{7} \end{array}$$

whence, subtracting the second line from the third in each case, we find

$$90x = 25, \qquad 9900y = 2433,$$

$$\text{and } x = \frac{25}{90} = \frac{5}{18} : \qquad \text{and } y = \frac{2433}{9900} = \frac{811}{3300} :$$

and these put in the following forms,

$$x = \frac{25}{90} = \frac{27-2}{90} \qquad y = \frac{2433}{9900} = \frac{2457-24}{9900},$$

furnish us with a general rule.

RULE. Make the non-recurring and the recurring parts taken *together*, diminished by the non-recurring part taken *alone*, the numerator of a fraction whose denominator shall consist of as many *nines* as there are recurring figures, followed by as many *ciphers* as there are non-recurring figures, and this reduced to its lowest terms will be the vulgar fraction required.

111. It will hence appear that the arithmetical operations upon recurring decimals, may be correctly effected by means of the same operations performed upon their equivalent vulgar fractions.

Ex. Let it be required to find the sum, difference, product and quotient, of the recurring decimals $.\dot{6}$ and $.\dot{2}9\dot{6}$.

Here, by the rules, we have $.\dot{6} = \frac{2}{3}$, and $.\dot{2}9\dot{6} = \frac{8}{27}$:

$x = .296$
$1000x = 296.296$
$999x = 296$
$x = \frac{296}{999}$
$= \frac{37 \times 8}{37 \times 27}$
$\therefore x = \frac{8}{27}$

$$\text{therefore, the sum } = \frac{2}{3} + \frac{8}{27} = \frac{26}{27} = .\dot{9}6\dot{2} :$$

$$\text{the difference } = \frac{2}{3} - \frac{8}{27} = \frac{10}{27} = .\dot{3}7\dot{0} :$$

$$\text{the product } = \frac{2}{3} \times \frac{8}{27} = \frac{16}{81} = .\dot{1}9753086\dot{4} :$$

$$\text{the quotient } = \frac{2}{3} \div \frac{8}{27} = \frac{9}{4} = 2.25 :$$

the first three of which are recurring decimals, and the last a finite quantity when expressed decimally: and it may be remarked that the same results could have been obtained by the *immediate* operations only by means of a laborious process.

112. In the same manner recurring decimals of *specified* units may be treated, and their exact values thence obtained.

Ex. Find the value of $.1\dot{6}$ of a pound sterling.

$$\text{Here, } .1\dot{6}\ £ = \frac{1}{6}\ £ = \frac{1}{6} \times \frac{20}{1}\ s. = \frac{20}{6}\ s. = 3\ s.\ 4\ d.$$

113. Since, in converting a vulgar fraction into a decimal, either 10, 100, 1000, &c., or their multiples, are divided by the denominator, it is evident that the decimal will *terminate* or *not*, according as these numbers *are* divisible by the denominator or *not:* whence, as the only incomposite factors of 10, 100, 1000, &c., are 2 and 5, it follows that vulgar fractions, whose denominators can be resolved into these factors, are equivalent to finite decimals, whilst all others are not.

Thus, $\frac{3}{50} = \frac{3}{2 \times 5 \times 5} = .06$, a finite decimal:

$\frac{5}{12} = \frac{5}{2 \times 2 \times 3} = .41\dot{6}$, a recurring decimal.

Examples for Practice.

(1) What are the recurring decimals corresponding to the vulgar fractions,

$$\frac{2}{9},\ \frac{3}{11},\ \frac{13}{99} \text{ and } \frac{129}{55}?$$

Answers: $.\dot{2}$, $.\dot{2}\dot{7}$, $.\dot{1}\dot{3}$ and $2.3\dot{4}\dot{5}$.

(2) Convert $\frac{4}{13}$, $\frac{5}{41}$ and $\frac{8}{53}$ into recurring decimals.

Answers: $.\dot{3}0769\dot{2}$, $.\dot{1}219\dot{5}$ and $.\dot{1}50943396226\dot{4}$.

(3) Find the vulgar fractions equivalent to the recurring decimals: $.\dot{5}$, $.\dot{0}2\dot{7}$ and $.\dot{5}3\dot{4}$.

Answers: $\frac{5}{9}$, $\frac{1}{37}$ and $\frac{178}{333}$.

(4) What vulgar fractions will represent the values of the recurring decimals, $.36\dot{2}\dot{1}$, $.47\dot{5}4\dot{3}$ and $.6\dot{7}6190\dot{4}$?

Answers: $\frac{239}{660}$, $\frac{3958}{8325}$ and $\frac{71}{105}$.

(5) Find the sum, difference, product and quotient of $.96\dot{3}4\dot{5}$ and $.\dot{3}$.

Answer:

the sum $= 1.29\dot{6}7\dot{8}$, the difference $= .63\dot{0}1\dot{2}$,
the ~~product~~ quotient $= .32\dot{1}1\dot{5}$, the ~~quotient~~ product $= 2.89\dot{0}3\dot{6}$.

(6) Reduce 9 oz. $2\frac{2}{7}$dr. to the decimal of 1 lb.

Answer: $.\dot{5}7142\dot{8}$.

(7) Find the values of $.97291\dot{6}$ of £1., and of $.013\dot{8}$ of 3.5 moidores.

Answers: 19*s*. $5\frac{1}{2}$*d*., and 1*s*. $3\frac{3}{4}$*d*.

(8) Required the exact value of .75 of 6*s*. 8*d*. $-$ 1.84375 of 4*s*. + $3.979\dot{6}$ of 2*s*.

Answer: 5*s*. 7.012*d*.

(9) The price of .0625 lbs. of coffee being $.458\dot{3}$*s*., what is the cost of .075 of a ton?

Answer: £61. 12*s*.

(10) If a vulgar fraction be converted into a recurring decimal, the number of figures which recur will always be less than its denominator.

CHAPTER VI.

RATIO AND PROPORTION,

WITH SOME OF THEIR MOST IMPORTANT APPLICATIONS.

RATIO.

114. Def. 1. *Ratio* is the relation which one number has to another, or, which one *quantity* numerically considered bears to another of the *same kind*, the comparison being made by observing what *multiple, part* or *parts*, the one is of the other.

Thus, the relation of 2 to 1, whereof the former is *double* of the latter, is regarded as the ratio of those numbers: and it is *written* 2 : 1, and usually *read* two to one.

115. Def. 2. Of the numbers or quantities compared and called the *Terms* of the ratio, the former is styled the *Antecedent*, and the latter the *Consequent;* also, the ratio is a ratio of *greater* or *less Inequality*, according as the antecedent is *greater* or *less* than the consequent; and it is a ratio of *Equality*, when those terms are *equal*.

Thus, the ratio 6 : 5 is a ratio of greater inequality; 4 : 9 is one of less inequality, and a ratio of equality may be denoted by 1 : 1, or 2 : 2, or 3 : 3, &c., at pleasure.

116. From the preceding definitions, it follows that the *Magnitude* of a ratio is expressed by the vulgar fraction, whereof the antecedent is the *Numerator*, and the consequent the *Denominator;* thus, the ratio of 9 and 12, written 9 : 12, will have its magnitude expressed by the fraction $\frac{9}{12}$, or, reduced to lower terms, by the fraction $\frac{3}{4}$.

Similarly, if the terms of the ratio be vulgar fractions or decimals, the fraction expressing its magnitude may be simplified by the rules already given.

117. The magnitudes of two or more ratios may therefore be compared, by comparing the values of the vulgar fractions which represent them, according to the principle of the last article.

If the ratios be $3:4$ and $5:7$; then will their magnitudes be represented by $\frac{3}{4}$ and $\frac{5}{7}$;

$$\text{but } \frac{3}{4}=\frac{21}{28} \text{ and } \frac{5}{7}=\frac{20}{28}, \text{ by article (85),}$$

where it is clear that $\frac{21}{28}$ is greater than $\frac{20}{28}$; whence it follows that the ratio $3:4$ is greater than the ratio $5:7$; in other words, 3 has to 4 a greater ratio than 5 has to 7.

118. *A ratio of greater inequality is diminished, and a ratio of less inequality is increased, by adding the same quantity to both its terms.*

First, let us take the ratio of greater inequality $7:5$, and add 1 to both its terms, so that it becomes $8:6$;

$$\text{then the former ratio} = \frac{7}{5} = \frac{42}{30},$$

$$\text{and the latter ratio} = \frac{8}{6} = \frac{40}{30};$$

from which it appears that the new ratio is less than the original one.

Secondly, taking the ratio of less inequality $8:11$, and adding 2 to each term, so as to make it $10:13$, we have

$$\text{the original ratio} = \frac{8}{11} = \frac{104}{143},$$

$$\text{and the new ratio} = \frac{10}{13} = \frac{110}{143};$$

the latter of which fractions being greater than the former, the new ratio is, of course, the greater of the two.

Exactly in the same manner may it be shewn that a ratio of greater inequality is increased, and a ratio of less inequality is diminished, by subtracting the same quantity from each of its terms.

119. *If the terms of a ratio be multiplied or divided by the same quantity, the magnitude of the ratio will not be altered.*

Let the ratio be $3 : 8$; then its magnitude is $\frac{3}{8}$, which is equivalent to

$$\frac{6}{16}, \text{ or } \frac{9}{24}, \text{ or } \frac{12}{32}, \text{ \&c.}:$$

that is, the ratio $3 : 8$ is equal to each of the ratios $6 : 16$, $9 : 24$, $12 : 32$, &c., which arise from the equal *multiplication* of its terms: and conversely, each of the latter ratios is reducible to the original one by equal *division* of its terms.

120. Def. 3. If the antecedents of two or more ratios be multiplied together for a new antecedent, and their consequents be multiplied together for a new consequent, the resulting ratio is said to be *compounded* of the others, and is sometimes called their *Sum*.

Thus, if there be proposed the three ratios,

$$2 : 3,\ 4 : 7 \text{ and } 8 : 13,$$

the ratio which arises from their composition will be

$$2 \times 4 \times 8 : 3 \times 7 \times 13, \text{ or } 64 : 273.$$

Examples for Practice.

(1) What are the simplest expressions of the magnitudes of the ratios $3 : 5$, $4 : 12$ and $9 : 21$?

Answers: $\frac{3}{5}$, $\frac{1}{3}$ and $\frac{3}{7}$.

(2) Which of the ratios is greater, $5 : 9$ or $7 : 11$; $10 : 17$ or $17 : 23$, and $34 : 27$ or $37 : 31$?

Answers: $7 : 11$, $17 : 23$, and $34 : 27$.

(3) Which of the three ratios $7 : 15$, $1\frac{3}{4} : 2\frac{5}{8}$, and $.75 : .96$ is the greatest?

Answer: $.75 : .96$.

(4) Find whether the ratios $7 : 9$, $11 : 17$ and $10 : 7$ are increased or diminished by adding 1, 2, 3, to their terms respectively.

Answer:

The first and second are increased, and the third is diminished.

(5) Are the ratios $4 : 3$, $9 : 13$ and $15 : 22$ increased or diminished by subtracting 2, 3, 4, from their terms respectively?

Answer:

The first is increased, and the second and third are diminished.

(6) What are the ratios arising from the composition of $5 : 12$ and $6 : 25$; and of $5 : 7$, $7 : 18$ and $18 : 35$?

Answers: $1 : 10$, and $1 : 7$.

PROPORTION.

121. Def. 1. *Proportion* is the relation of *Equality* subsisting between *two* or *more* ratios.

Thus, the two ratios $2 : 3$ and $6 : 9$, being expressible by the fractions $\frac{2}{3}$ and $\frac{6}{9}$, are *equal*, and the four numbers 2, 3, 6, 9 form a proportion which is *written*

$$2 : 3 = 6 : 9, \text{ or } 2 : 3 :: 6 : 9,$$

and is *read*

as 2 is to 3, so is 6 to 9,

the numbers 2, 3, 6, 9 being called its *Terms*.

Hence, in every proportion, the first term is greater than, equal to, or less than the second, according as the third term is greater than, equal to, or less than the fourth.

122. Def. 2. In a proportion thus expressed, the numbers 2 and 9 are called the *Extremes*, and the numbers 3 and 6 the *Means:* and it follows immediately from the equality of the ratios denoted by

$$\frac{2}{3} = \frac{6}{9},$$

and the multiplication of them both by 27, that

$$\frac{2}{3} \times 27 = \frac{6}{9} \times 27;$$

$$\text{that is, } 2 \times 9 = 6 \times 3:$$

or, in other words, if *four* numbers constitute a proportion, the product of the *extremes* is equal to the product of the *means*.

123. The property of a proportion stated in the last article, proves immediately that *either* of the extremes may be obtained, by dividing the product of the means by the *other;* and that *either* of the means may be had by the division of the product of the extremes by the *other:* also, these qualities constitute the general practical application of Proportion.

124. From what has already been said upon this subject, it is evident that the terms may be made to undergo changes and modifications in the same way as the *corresponding* terms of the vulgar fractions.

Thus, if four numbers form a proportion, and any equi-multiples whatever of the first and second be taken, and also any equi-multiples whatever of the third and fourth, the resulting numbers taken in order will still form a proportion.

For, since $5 : 3 :: 15 : 9$, or $\frac{5}{3} = \frac{15}{9}$; and also, $\frac{2}{2} = \frac{3}{3}$;

$$\text{we have } \frac{5}{3} \times \frac{2}{2} = \frac{15}{9} \times \frac{3}{3}, \text{ or } \frac{5 \times 2}{3 \times 2} = \frac{15 \times 3}{9 \times 3};$$

$$\text{whence, } 5 \times 2 : 3 \times 2 :: 15 \times 3 : 9 \times 3;$$

and the converse will evidently be true.

Again, if any equi-multiples whatever of the first and third numbers be taken, and also any equi-multiples whatever of the second and fourth, the numbers thence arising will form a proportion.

Thus, if we take the proportion above, we have

$$\frac{5}{3} \times \frac{4}{7} = \frac{15}{9} \times \frac{4}{7}, \text{ or } \frac{5 \times 4}{3 \times 7} = \frac{15 \times 4}{9 \times 7};$$

whence, $5 \times 4 : 3 \times 7 :: 15 \times 4 : 9 \times 7$; and conversely.

The division of the terms of a proportion, in accordance with this article, will often facilitate practical computations, by diminishing the number of figures necessary to be employed.

125. Of two or more proportions, if the corresponding terms be multiplied together, the numbers thence arising will also form a proportion.

Thus, if the proportions be

$$3 : 7 :: 6 : 14, \text{ and } 4 : 9 :: 12 : 27;$$

$$\text{then } \frac{3}{7} = \frac{6}{14}, \text{ and } \frac{4}{9} = \frac{12}{27};$$

$$\text{whence, } \frac{3}{7} \times \frac{4}{9} = \frac{6}{14} \times \frac{12}{27}, \text{ or } \frac{3 \times 4}{7 \times 9} = \frac{6 \times 12}{14 \times 27};$$

$$\text{and } 3 \times 4 : 7 \times 9 :: 6 \times 12 : 14 \times 27.$$

This operation is called the *Compounding* of proportions, and the last proportion is said to be *compounded* of the two others.

126. The terms Ratio and Proportion as here used, are generally called *Geometrical* Ratio and *Geometrical* Proportion, because they are employed in *Geometry* in the same sense: also, *Arithmetical* Ratio and *Arithmetical* Proportion are *sometimes* used to express the *Differences* of two or more numbers, and their relations to each other, exactly in the same manner as we have throughout applied Ratio and Proportion to denote their *Quotients*, and the relations subsisting among two or more such.

Thus, of 7 and 5, the *geometrical* ratio is $7 : 5 = \frac{7}{5}$; whereas their *arithmetical* ratio is $7 - 5 = 2$: also, the numbers 3, 4, 15, 20 form a *geometrical* proportion, because $\frac{3}{4} = \frac{15}{20}$: but 4, 3, 2, 1 constitute an *arithmetical* proportion, since $4 - 3 = 2 - 1$.

When necessary, the relations of numbers, considered in the latter point of view, may be determined by means of the equality $4 - 3 = 2 - 1$, in a manner similar to what has been done above.

127. If three numbers as 18, 13 and 8 be in what is called *continued* Arithmetical proportion, then $18 - 13 = 13 - 8$; and if $13 + 8$ be added to both members of this equality, we shall have

$$18 + 8 = 13 + 13;$$

that is, the *Sum* of the *Extremes* is equal to *twice* the *Arithmetical Mean* between them; and therefore the arithmetical mean is equal to *half* their sum.

In the same manner 16, 8 and 4 are said to be in *continued* Geometrical proportion, because

$$16 : 8 :: 8 : 4, \text{ or } \frac{16}{8} = \frac{8}{4};$$

and multiplying both sides of this equality by 8×4, we obtain

$$16 \times 4 = 8 \times 8,$$

or, the *Product* of the *Extremes* is equal to the *Square* of the *Geometrical Mean* between them: and consequently the geometrical mean between two numbers is equal to the *Square Root* of their product.

These *terms* and the corresponding *operations* form the substance of the next Chapter, and they have been noticed in this, only because they appear to arise immediately out of what has been considered in it.

Applications of Ratio and Proportion.

128. The subjects of most importance in a practical point of view, to which the doctrines of Ratio and Proportion are immediately applicable, seem to be the following:

(1) *The Rule of Proportion.*
(2) *Interest, Stocks, &c.*
(3) *Discount or Rebate.*
(4) *Equation of Payments.*
(5) *The Rule of Fellowship.*
(6) *The Rule of Alligation.*
(7) *The Doctrine of Exchanges.*
(8) *Miscellaneous Questions:*

and the principles of each will be explained and exemplified in the order in which they here stand.

I. THE RULE OF PROPORTION.

129. Def. As has been observed in *The Rule of Three,* of which this is merely another name, we have here *three* quantities, either simple or compound, given to find a *fourth,* which shall complete the proportion; and this quantity is called a *fourth proportional* to the three quantities proposed.

130. Assuming it as an axiom, that *Effects* have the same relation or ratio to each other, as the *Causes* which produce them under the same circumstances, it is evident, that in any two cases of the same kind, we shall have the following proportion:

First Cause : Second Cause :: First Effect : Second Effect;

and then, what was said in articles (122) and (123)

will enable us to find the magnitude of any *one* term, if those of the *three* others be given, and thus to solve the question.

To avoid the trouble of writing the name of the *required* term or quantity at length, we shall always denote it by the simple *symbol* x, which must be treated in the same way as any other number: also, it will not be necessary that this symbol should be the *fourth* term of the proportion, but it may occupy *any* situation either by *itself*, or in *connection* with given numbers, as will be manifest from the following examples.

Ex. 1. If 5 men can mow 12 acres of grass in a certain time; how many acres will 16 men be able to mow in the same or an equal time?

$$\text{Here, } \left.\begin{matrix} 5 \text{ men} \\ 16 \text{ men} \end{matrix}\right\} \text{ are the first and second } \left\{ \textit{Causes:} \right.$$

$$\left.\begin{matrix} 12 \text{ acres} \\ x \text{ acres} \end{matrix}\right\} \text{ are the first and second } \left\{ \textit{Effects:} \right.$$

whence we have the following proportion:

$$\overset{\text{men.}}{5} : \overset{\text{men.}}{16} :: \overset{\text{ac.}}{12} : \overset{\text{ac.}}{x};$$

and therefore by the articles just referred to,

$$5 \times x = 16 \times 12 = 192:$$

$$\text{whence, } x = \frac{\overset{\text{ac.}}{192}}{5} = \overset{\text{ac.}}{38} . \overset{\text{ro.}}{1} . \overset{\text{po.}}{24}.$$

Ex. 2. If 8oz. of bread be sold for 6*d*., when wheat is at £15. a load; what should be the price of wheat when 12oz. are sold for 4*d*.?

It is evident that the price of a load of wheat will be *regulated* by, and be *proportional* to, the price of an ounce of bread:

$$\text{now, in the former case, the price of } \overset{\text{oz.}}{1} = \frac{\overset{d.}{6}}{8} = \frac{\overset{d.}{3}}{4}:$$

$$\text{and in the latter, the price of } \overset{\text{oz.}}{1} = \frac{\overset{d.}{4}}{12} = \frac{\overset{d.}{1}}{3}:$$

therefore, as before, we have the following proportion,

$$\overset{d.}{\frac{3}{4}} : \overset{d.}{\frac{1}{3}} :: \overset{£.}{15} : \overset{£.}{x};$$

whence,

$$x = \frac{1}{3} \times 15 \div \frac{3}{4} = \frac{1}{3} \times \frac{15}{1} \times \frac{4}{3} = \overset{£.}{\frac{60}{9}} = \overset{£.}{\frac{20}{3}} = \overset{£.}{6} \,.\, \overset{s.}{13} \,.\, \overset{d.}{4},$$

the required price.

In both these examples, the causes are *simple* terms, being dependent upon only *one* magnitude.

Ex. 3. If 10 men can perform a piece of work in 12 days; how many days will it take 8 men to do the same?

Here, the causes will evidently be to each other, as 10×12 to $8 \times x$; and the effects are the *same*, and may therefore be represented by 1, or any other symbol:

$$\text{whence, } 10 \times 12 : 8 \times x :: 1 : 1;$$

$$\text{therefore } 8 \times x = 10 \times 12 = 120,$$

$$\text{and } x = \frac{120}{8} = 15 \text{ days.}$$

Ex. 4. How much in length, that is 3ft. 9in. broad, will be equivalent to what is 37ft. 9in. long, and 7ft. 6in. broad?

Here, by reasoning as before, we have

$$\text{the first cause} = \overset{\text{in.}}{45} \times \overset{\text{in.}}{x};$$

$$\text{the second cause} = 90 \times 453;$$

and the effects are to be equal:

$$\text{therefore } 45 \times x : 90 \times 453 :: 1 : 1;$$

$$\text{whence, } 45 \times x = 90 \times 453,$$

$$\text{and } x = \frac{90 \times 453}{45} = \overset{\text{in.}}{906} = \overset{\text{ft.}}{75} \,.\, \overset{\text{in.}}{6}.$$

In these two examples, the causes are *compound* quantities, depending upon two *subordinate* causes.

Ex. 5. If a person can go a journey of 100 miles in 12 days of 8 hours each; how far will he be able to travel in 15 days of 9 hours each?

Here, 12×8 and 15×9 are the causes, and the distances travelled 100 and x are the effects: whence,

$$12 \times 8 : 15 \times 9 :: 100 : x;$$

$$\text{and } x = \frac{15 \times 9 \times 100}{12 \times 8} = 140\tfrac{5}{8} \text{ miles.}$$

Ex. 6. If 60 bushels of corn feed 6 horses for 50 days; in how many days will 15 horses consume 75 bushels?

The causes are 6×50 and $15 \times x$, and the effects are 60 and 75 bushels: and therefore

$$6 \times 50 : 15 \times x :: 60 : 75,$$

$$\text{or, } 2 \times 10 : x :: 4 : 5;$$

$$\text{whence, } x = \frac{2 \times 10 \times 5}{4} = 25 \text{ days.}$$

In the former of these examples, the distances travelled are in the *compound* ratio of the numbers of days and their lengths: and in the latter, the numbers of bushels of corn have the same ratio as that which is *compounded* of the numbers of horses and days.

Ex. 7. If 25 labourers can dig a trench 220 yards long, 3ft. 4in. wide, and 2ft. 6in. deep, in 32 days of 9 hours each: how many would it require to dig a trench half a mile long, 2ft. 4in. deep, and 3ft. 6in. wide, in 36 days of 8 hours each?

$$\left.\begin{array}{l}\text{First cause} = 25 \times 32 \times 9 \\ \text{second cause} = x \times 36 \times 8\end{array}\right\} \text{being the products of the subordinate causes:}$$

$$\left.\begin{array}{l}\text{first effect} = 220 \times \frac{10}{9} \times \frac{5}{6} \\ \text{second effect} = 880 \times \frac{7}{9} \times \frac{7}{6}\end{array}\right\} \text{the mixed quantities being reduced to fractions of 1 yard.}$$

Hence, we have the following proportion:

$$25 \times 32 \times 9 : x \times 36 \times 8 :: 220 \times \frac{10}{9} \times \frac{5}{6} : 880 \times \frac{7}{9} \times \frac{7}{6};$$

$$\text{or, } 25 : x :: 1 \times 10 \times 5 : 4 \times 7 \times 7;$$

$$\text{whence, } x = \frac{25 \times 4 \times 7 \times 7}{1 \times 10 \times 5} = 98 \text{ labourers.}$$

In this example, both the causes and effects are compound quantities, consisting of their respective subordinate *partial* causes and effects.

131. In practice, when the *partial* causes and effects consist of compound quantities, it is most convenient to express them by vulgar fractions or decimals: and when the *entire* causes and effects are compound quantities, to proceed as in the third chapter, shortening the operations as much as possible by means of article (124).

Examples for Practice.

(1) Find a number which shall have the same ratio to 7 that 27 has to 3: also, a magnitude to which 39 has the same ratio as $3\frac{1}{4}$ has to $2\frac{2}{5}$.

Answers: 63 and $31\frac{1}{5}$.

(2) Find the price of 39 cwt. 3 qrs. 26 lbs. at £4. 17*s*. 10*d*. per cwt.

Answer: £195. 11*s*. $7\frac{1}{28}$*d*.

(3) What quantity of cloth at 6*s*. 8*d*. a yard may be bought for 20 guineas?

Answer: 63 yards.

(4) If a piece of cloth measuring 9 ells. 1 na. $1\frac{1}{8}$ in., cost £3. 15*s*. $7\frac{1}{2}$*d*., what is its price per yard?

Answer: 6*s*. 8*d*.

(5) How much carpet 2 ft. 3 in. wide, will cover a floor 13 ft. 6 in. long, and 10 ft. wide?

Answer: 20 yards.

(6) What is the price of 19 cwt. 2 qrs. 23 lbs., when 4 cwt. 1 qr. cost £3. 14*s*. $6\frac{1}{2}$*d*.?

Answer: £17. 5*s*. $7\frac{1}{4}$*d*.$\frac{141}{238}$.

(7) The rental of a parish is £5497. 13*s*. 4*d*., and £152. 10*s*. 6*d*. is to be raised by a rate: what is the rate in the pound?

Answer: $6\frac{1}{2}$*d*.$\frac{10454}{18493}$.

(8) If a person can perform a journey in 24 days of $10\frac{1}{3}$ hours each; what time will it take him to do the same when the days are $12\frac{3}{4}$ hours long?

Answer: $19\frac{13}{17}$ days.

(9) How much in length, that is 15 poles in breadth, will be equivalent to an acre of land, which is 40 poles in length and 4 poles in breadth?

Answer: 10po. 3yds. 2ft.

(10) If £100. be sufficient to discharge a debt of £104. 17*s*. 6*d*. due a year hence; how much money will be sufficient to pay a debt of £1000. at the same date?

Answer: £953. 10*s*. $3\frac{3}{4}$*d*.$\frac{375}{839}$.

(11) If in 365 days. 5 hrs. 49 min., the Sun describe an arc of 360° in the heavens: what is his mean daily motion?

Answer: 59′. 8.328 &c.″

(12) If 7 men earn £9. 10*s*. 6*d*. in $10\frac{1}{2}$ days; what sum will 28 men earn in $31\frac{1}{2}$ days?

Answer: £114. 6*s*.

(13) If with a capital of £500. a tradesman gain £100. in 14 months; in what time will he gain £60. 10*s*. with a capital of £770?

Answer: $5\frac{1}{2}$ months.

(14) If 400 soldiers consume 5 barrels of flour in 12 days; how many soldiers will consume 15 barrels in 2 days?

Answer: 7200 soldiers.

(15) If 20 men can perform a piece of work in 12 days; how many men will perform another piece of work three times as great, in a fifth part of the time?

Answer: 300 men.

(16) If 12 men can mow a field 300 yards square in 10 days: how many men can mow a field 600 yards long, and 10 yards wide, in 4 days?

Answer: 2 men.

(17) A bankrupt owes £1490. 5*s*. 10*d*., and has only £784. 17*s*. 4*d*.; how much will his creditors receive in the pound?

Answer: 10*s*. $6\frac{1}{4}$*d*.$\frac{20998}{35767}$.

(18) If 27 men can do a piece of work in 14 days, working 10 hours a day; how many hours a day must 24 boys work, in order to complete the same in 45 days, the work of a boy being half that of a man?

Answer: 8 hours.

(19) If 10 cannon, which fire 3 rounds in 5 minutes, kill 270 men in $1\frac{1}{2}$ hours; how many cannon, which fire 5 rounds in 6 minutes, will kill 500 men in 1 hour, at the same rate?

Answer: 20 cannon.

(20) If 120 men in 3 days of 12 hours each, can dig a trench 30 yds. long, 2 ft. broad, and 4 ft. deep; how many men would be required to dig a trench 50 yds. long, 6 ft. deep, and $1\frac{1}{2}$ yds. broad, in 9 days of 15 hours each?

Answer: 180 men.

(21) A watch, which is 10 minutes too fast at twelve o'clock on Monday, gains 3 min. 10 sec. per day; what will be the time by the watch at a quarter past ten in the morning of the following Saturday?

Answer: 40 min. $36\frac{7}{48}$ sec. past 10.

(22) Find a fourth proportional to 35, $\frac{1}{20}$ and $3\frac{3}{4}$: also, to 125, .0145 and .35.

Answers: $\frac{3}{560}$ and .0000406.

(23) How much cloth $\frac{1}{2}$ yard wide, will cover a room 12 ft. 6 in. long, and 2 ft. 9 in. wide; and what will it cost at 5*s.* 6*d.* a yard?

Answers: 7 yds. $2\frac{5}{9}$ qrs., and £2. 2*s.* $0\frac{1}{6}$*d.*

(24) If beer, which is brewed with 3 bushels of malt to the barrel, cost 1*s.* 3*d.* per gallon, when malt is at 62*s.* 8*d.* the quarter: how much will beer cost per gallon, which is brewed with 5 bushels of malt to the barrel, when a quarter of malt costs 50*s.*?

Answer: 1*s.* $7\frac{3}{4}$*d.* $\frac{37}{47}$.

II. INTEREST, STOCKS, &c.

132. DEF. *Interest* is the payment made for the loan or use of money for any length of time, being generally estimated at so much for £100. during a year, and commonly expressed by so much *per cent. per annum:* the money lent is called the *Principal,* the interest of £100. for a year, the *Rate per cent.*: and the sum lent together with its interest is termed the *Amount.*

It is called *simple* interest, when the loan itself only pays interest for the whole time it is lent; and *compound*

interest, when, at the end of any *assigned* period, as a year for instance, the interest, which has accrued, is added to the principal, and the whole then bears interest at the same rate for another *equal* period, and so on.

Hence, it is evident that for one year, the sum lent may be regarded as the *cause*, and the interest produced as the *effect*.

Simple Interest.

Ex. Find the simple interest and amount of £237. 10*s*. for $2\frac{1}{2}$ years, at 5 per cent. per annum.

From what has just been said, we have

$$\overset{\pounds.}{100} : \overset{\pounds.}{237} . \overset{s.}{10} :: \overset{\pounds.}{5} : \overset{\pounds.}{x};$$

and therefore

$$x = \frac{\overset{\pounds.}{237} . \overset{s.}{10} \times 5}{100} = \frac{\overset{\pounds.}{237.5} \times 5}{100} = \frac{\overset{\pounds.}{1187.5}}{100} = \overset{\pounds.}{11.875} = \overset{\pounds.}{11} . \overset{s.}{17} . \overset{d.}{6}$$

is the interest of $\overset{\pounds.}{237} . \overset{s.}{10}$. for one year:

hence, $\overset{\pounds.}{11} . \overset{s.}{17} . \overset{d.}{6} \times 2\frac{1}{2} = \overset{\pounds.}{29} . \overset{s.}{13} . \overset{d.}{9}$ is the interest of $\overset{\pounds.}{237} . \overset{s.}{10}$ for $2\frac{1}{2}$ years: and therefore

$$\overset{\pounds.}{29} . \overset{s.}{13} . \overset{d.}{9} + \overset{\pounds.}{237} . \overset{s.}{10} = \overset{\pounds.}{267} . \overset{s.}{3} . \overset{d.}{9}$$

is the amount of the same sum for the same time.

In practice, we may work in the subsequent form, following up the same principles: thus,

$$\begin{array}{r}
\pounds. \quad s. \\
237 . 10 \\
5 \\
\hline
\pounds\, 11 . 87 . 10 \\
20 \\
\hline
s\, 17 . 50 \\
12 \\
\hline
d\, 6 . 00 \\
\hline
\end{array}$$

where the sum proposed is multiplied by the rate per cent., and from the right of each successive denomination, *two* figures are cut off instead of dividing by 100, according to the principles of decimals: whence, we have

	£.	s.	d.	
	11	17	6	= interest for 1 year:
			$2\frac{1}{2}$	
	23	15	0	= interest for 2 years:
	5	18	9	= interest for $\frac{1}{2}$ year:
	29	13	9	= interest for $2\frac{1}{2}$ years:
	237	10	0	= principal:
£	267	3	9	= amount in $2\frac{1}{2}$ years:

and this form gives rise to the following rule.

Rule for Simple Interest.

Multiply the principal by the rate per cent. by Compound Multiplication or Practice: from the pounds in the product cut off *two* figures to the *right*, and the remaining figures will be the pounds of the interest: reduce the figures cut off to shillings, taking in the shillings of the said product, cut off as before, and thus proceed; and the interest for one year will be obtained: multiply this interest by the number of years proposed, whether whole or fractional, and the required interest will be found.

When the interest for *months* and *days* is required, it is usually found by *Practice* or the *Rule of Three* respectively, reckoning 12 months, and 365 days to a year: but if *calendar* months be specified, the interest is more accurately determined by finding the number of days they contain, and proceeding according to the Rule of Three.

133. *Commission, Brokerage, Insurance, &c.*, being charges of certain sums *per cent.*, manifestly amount to the same thing as the interest for one year at the same rate, and they may therefore be computed by the same rule.

Compound Interest.

Ex. Required the compound interest of £250. for two years, at 5 per cent.

```
       £.
      250
        5
   £12.50
       20
   s10.00

£.      s.
250 .   0 = first principal:
 12 .  10 = interest for the first year:
£262 . 10 = amount in one year:

      £.    s.
     262 . 10
            5
£13 . 12 . 10
       20
     s 2.50
       12
     d 6.00

 £.   s.   d.
262 . 10 . 0 = second principal:
 13 .  2 . 6 = interest for the second year:
275 . 12 . 6 = amount in two years:
250 .  0 . 0 = original principal:
£ 25 . 12 . 6 = compound interest in two years:
```

and it is easily seen that this interest is the *sum* of the interests of the *first* and *second* years together, upon their respective principals.

Here the interest may be supposed to form part of the new principal at the ends of any *equal* intervals of time, as *half-yearly*, *quarterly*, &c.; but then the operation above must be repeated for *every* such interval; and when the compound interest is required for any

number of years and parts of a year, it is inconsistent with the principles of the subject, to suppose that the interest becomes due at any other interval of time than what is expressed by the *primitive* fraction of which the parts are made up: as for instance, when compound interest for *three-fourths* of a year is required, it is necessarily implied that the interest is due at the end of each *quarter*.

When the *equitable* principle just mentioned is not attended to, it is *customary* to find the interest for one year in addition to the number of entire years expressed, and then to take the part or parts of that interest which correspond with the proposed part or parts of a year, and to add it to the amount already obtained: but although this be not a bad approximation to the true amount, questions for the *exercise* of students should never be proposed which require its application.

The operation last given, will preclude the necessity of laying down a rule in *words*, for finding the amount and compound interest of any sum, as far as practice may be concerned.

The Natures and Transfers of Stocks.

134. DEF. The exigencies of a Country sometimes compel its governing body to *borrow*, or to *contract a Loan*, for the benefit of the public service: and this is effected, by giving to the *Lenders* in exchange for their money, *Government Bonds* or *Acknowledgments*, implying that the Nation is indebted to them for the sums advanced, whilst it reserves to itself the option of the *Time* of paying off the *Principal*, on the express condition that the *Interest* is regularly discharged at the time fixed upon.

135. Any part of these bonds is *transferable* from one person to another at pleasure, and each bond is usually styled £100. *Stock*, bearing interest at a certain rate, the subdivisions of £1. stock, being the same as those of *sterling* money.

Thus, in what are called the 3, $3\frac{1}{2}$, and 4 *per cent.* Stocks, one of these bonds entitles its owner to the sums of £1. 10*s*., £1. 15*s*., and £2. respectively, at the end of every *half-year*, as interest: and that portion of the revenues of the country out of which the interest of these *Stocks* and the expenses connected with them are paid, is termed the *Funds*.

136. If a person *sell out* his stock from the Funds, he will be able to obtain more or less sterling money for each of his bonds, according to the interest it bears, and also according to the circumstances of the times, which may influence the *stability* of the national credit: and if he *buy into,* or *invest capital* in the Funds, the sum of ready money advanced by him for each bond, will of course be regulated by the same circumstances.

137. When a *transfer* of capital is made from one kind of stock to another, it is evident that there will be an equitable claim for *more* or *fewer* bonds of the second stock, according as the rate of interest of such bonds is *less* or *greater* than that of the first: thus, a number of bonds, or *quantity of stock* in the 4 per cents., will produce the same interest as a *greater* quantity of stock in the 3 per cents., and consequently be of the same value to the possessor, in point of income.

138. From this view of the subject, it follows that the computations necessary in all equitable transactions in the Stocks, must depend upon the *Rule of Proportion,* or the *Golden Rule:* and those of most frequent occurrence will be explained in the subsequent examples.

Ex. 1. How much money must be paid for £2400. in the three per cent. *consols* (consolidated annuities), at 89½ per cent.?

Here, we have the following proportion:

$$\begin{array}{ccccccc} \pounds.\text{ stock.} & & \pounds.\text{ stock.} & & \pounds. & & \pounds. \\ 100 & : & 2400 & :: & 89\tfrac{1}{2} & : & x; \end{array}$$

and the operation may be conducted as below:

$$\begin{array}{rl} \pounds\ 2400 & \\ 10 & \times 9 - \tfrac{1}{2} = 89\tfrac{1}{2} \\ \hline 24000 & \\ 9 & \\ \hline 2{\cdot}16000 & \\ 1200 & \\ \hline \pounds\ 2148.00 & \end{array}$$

that is, £2148. *sterling* will purchase £2400. of *this* stock, when it is at 89½ per cent.

If we reverse the operation, we may find the quantity of stock at $89\frac{1}{2}$, which may be purchased for £2148. sterling, as follows:

$$\overset{\pounds.}{89\tfrac{1}{2}} : \overset{\pounds.}{2148} :: \overset{\pounds.\text{ stock.}}{100} : \overset{\pounds.\text{ stock.}}{x};$$

$$\text{and therefore } x = \frac{2148 \times 100}{89\frac{1}{2}} = \pounds 2400.$$

Ex. 2. A person invests £3000. in the three per cents. when they are at $90\frac{1}{5}$; what amount of interest will he receive half yearly?

Here, £$90\frac{1}{5}$ sterling produces £3. yearly, or £$1\frac{1}{2}$. half-yearly: and therefore we have

$$\overset{\pounds.}{90\tfrac{1}{5}} : \overset{\pounds.}{3000} :: \overset{\pounds.}{1\tfrac{1}{2}} : \overset{\pounds.}{x};$$

$$\text{whence, } x = \frac{3000 \times 1\frac{1}{2}}{90\frac{1}{5}} = \frac{3000 \times 15}{902}$$

$$= \frac{1500 \times 15}{451} = \frac{22500}{451} = \pounds 49.\ 17s.\ 9\tfrac{1}{4}d.\ \tfrac{257}{451},$$

is his half-yearly dividend.

Hence, conversely, if a person wish to receive the last-mentioned sum half-yearly, he may invest £3000. in the three per cents. when they are at $90\frac{1}{5}$, for this purpose; since

$$\overset{\pounds.}{1\tfrac{1}{2}} : \overset{\pounds.}{49} \,.\, \overset{s.}{17} \,.\, \overset{d.}{9\tfrac{1}{4}}.\tfrac{257}{451} :: \overset{\pounds.}{90\tfrac{1}{5}} : \overset{\pounds.}{3000};$$

and of course the same income might be acquired by the investment of a different sum of money in a different kind of stock, dependent upon the circumstances affecting it.

Ex. 3. At what rate will a person receive interest, who invests his capital in the 4 per cents. when they are at 104?

Since £104. sterling produces an interest of £4. annually, we have

$$\overset{\pounds.}{104} : \overset{\pounds.}{100} :: \overset{\pounds}{4} : x$$

and therefore

$$x = \frac{\overset{\pounds.}{100 \times 4}}{104} = \frac{\overset{\pounds.}{50}}{13} = \pounds 3.\ 16s.\ 11\tfrac{1}{13}d. \text{ is the rate per cent.;}$$

and conversely, when the interest of money is £3. 16*s*. 11$\frac{1}{13}$*d*. per cent., the equitable value of the 4 per cent. stock is £104. sterling.

Ex. 4. A person transfers £1000. stock from the 4 per cents. at 90, to the 3 per cents. at 72: find how much of the latter stock he will hold, and the alteration made in his annual income.

Here, x denoting the quantity of the latter stock, we have

$$1000 \times 90 : x \times 72 :: 1 : 1;$$

$$\text{whence, } x = \frac{1000 \times 90}{72} = \frac{10000}{8} = \pounds 1250,$$

is the quantity of stock in the three per cents.:
also, £1000. at 4 per cent. gives an income of £40.:
and £1250. at 3 per cent. gives an income of £37. 10*s*.:
and therefore, the diminution of income is £2. 10*s*.

From this example, it appears to be *more advantageous* to invest in the 4 per cents. at 90, than in the 3 per cents. at 72, as will be seen also from the circumstance of $\frac{4}{90}$ being *greater* than $\frac{3}{72}$, which shews the reason at once.

139. In accordance with the principles employed in these examples, all transactions and dealings in the Stocks will be *equitably* conducted.

140. *Purchases* and *Sales* of stock are usually made through *Agents*, called *Stock-Brokers*, at the rate of £$\frac{1}{8}$ or 2*s*. 6*d*. per cent. upon the stock *transferred:* these agents are employed by both *buyers* and *sellers*, and the *brokerage* must therefore be *added* to the price of stock which is *bought*, and *subtracted* from the price of that which is *sold*, through them: and this is generally done by *increasing* or *diminishing* the current price of £100. stock by £$\frac{1}{8}$.

141. *Stock-jobbing* is dealing in the Stocks with the view of gaining money, by the rise and fall of the

market price: and it seems difficult to say how far a person may not justly take advantage of these circumstances for his own benefit, provided he be willing and able to answer all the demands which may be made upon him, in consequence of the risk and hazard of such speculations.

Examples for Practice.

(1) Find the simple interest of £382. 10*s*. for 1 year, at 5 per cent.

Answer: £19. 2*s*. 6*d*.

(2) What is the amount of £345. 17*s*. 6*d*. in 3 years, at 4 per cent. simple interest?

Answer: £387. 7*s*. $7\frac{1}{5}$*d*.

(3) Required the amount of £537. 16*s*. 8*d*. in 4 years, at $2\frac{1}{2}$ per cent. simple interest.

Answer: £591. 12*s*. 4*d*.

(4) Determine the amount of £635. 18*s*. $4\frac{1}{2}$*d*. in $3\frac{1}{2}$ years, at 3 per cent. simple interest.

Answer: £702. 13*s*. $9\frac{1}{2}$*d*. $\frac{61}{100}$.

(5) Find the amount of £325. 16*s*. 8*d*. at $4\frac{1}{4}$ per cent. simple interest, in $3\frac{1}{2}$ years.

Answer: £374. 6*s*. $0\frac{1}{4}$*d*.

(6) Required the amount of £825. 13*s*. 8*d*. at $4\frac{3}{4}$ per cent. simple interest, in 3 years and 5 months.

Answer: £959. 13*s*. $8\frac{1}{4}$*d*. $\frac{189}{960}$.

(7) What is the interest of £535. for 117 days, at $4\frac{3}{4}$ per cent.?

Answer: £8. 2*s*. $11\frac{8}{365}$*d*.

(8) Find the simple interest of £960. 12*s*. 6*d*. for 5 years, 8 months and 73 days, at $3\frac{1}{4}$ per cent.

Answer: £183. 3*s*. $2\frac{1}{5}$*d*.

(9) What is the interest of £240. from January 7, to September 12, 1839, at 4 per cent.?

Answer: £6. 10*s*. $5\frac{1}{4}$*d*. $\frac{308}{365}$.

(10) Find the amount of £237. 10*s*. for 2 years 8 months and 29 days, at 5 per cent. simple interest.

Answer: £270. 2*s*. $2\frac{1}{4}$*d*. $\frac{55}{73}$.

(11) Required the amount of £350. in 3 years, at 5 per cent. compound interest.

Answer: £405. 3*s*. 4½*d*.

(12) Find the compound interest of £540. in 2 years, at 4 per cent.

φ £21..12

Answer: ~~£44. 1*s*. 3¼*d*. $\frac{11}{25}$.~~ = 22..9..3 + $\frac{9}{25}$

(13) What is the compound interest of £150. in 4 years, at 2½ per cent.?

Answer: £15. 11*s*. 5¼*d*. $\frac{9}{160}$.

(14) What is the purchase of £5050. stock, at 85⅜ per cent.?

Answer: £4311. 8*s*. 9*d*.

(15) If the 4 per cents. be at 82⅛, what quantity of stock can be purchased for £821. 5*s*.?

Answer: £1000.

(16) A person invests £2000. in the 3 per cent. consols, when they are at 88⅜: what annual income is he entitled to?

Answer: £67. 15*s*. 11$\frac{11}{35}$*d*.

(17) How much stock can be purchased by the transfer of £1000. stock from the 3 per cents. at 72, to the 4 per cents. at 90; and what annual income will it produce?

Answer: £800, and £32.

(18) If I buy £650. stock in the 3 per cents. at 90⅜, and pay ⅛ for brokerage: what does it cost me?

Answer: £588. 5*s*.

(19) What sterling money shall I receive for £1760. 16*s*. 8*d*. stock at 90⅜, and ⅛ per cent. commission?

Answer: £1589. ~~8*s*.~~ 0½*d*.

(20) How much stock in the 3 per cents. will £1490. purchase; the price of stocks being 88⅜, and brokerage 2*s*. 6*d*. per cent.?

Answer: £1683. 12*s*. 3¾*d*. $\frac{11}{35}$.

φ. obs. the answer in the book is Right. & it should not be dashed over.

III. DISCOUNT OR REBATE.

142. Def. *Discount* or *Rebate* is an allowance or abatement made upon a debt discharged before it is due, at a certain rate per cent. in consideration of ready money: and when the discount is subtracted from any proposed sum, the remainder is termed the *Present Worth*.

Ex. Required the Present Worth and Discount of £275. 6*s*. 8*d*. due 18 months hence, at $4\frac{1}{2}$ per cent.

Since £100. at $4\frac{1}{2}$ per cent. amounts to £106. 15*s*. in 18 months, or $1\frac{1}{2}$ year; it is evident that £106. 15*s*. due 18 months hence, is of the same value to the owner as £100. ready money: and therefore we have

$$\overset{\pounds.}{106} \,.\, \overset{s.}{15} : \overset{\pounds.}{275} \,.\, \overset{s.}{6} \,.\, \overset{d.}{8} :: \overset{\pounds.}{100} : \overset{\pounds.}{x};$$

whence,

$$x = \frac{\overset{\pounds.}{275\frac{1}{3}} \times 100}{106\frac{3}{4}} = \frac{\overset{\pounds.}{3304} \times 100}{1281} = \frac{\overset{\pounds.}{330400}}{1281} = \overset{\pounds.}{257} \,.\, \overset{s.}{18} \,.\, \overset{d.}{5\frac{1}{2}}.\tfrac{34}{61}$$

is the required present worth:

and the discount = the proposed sum − the present worth

$$\text{is } \overset{\pounds.}{275} \,.\, \overset{s.}{6} \,.\, \overset{d.}{8} - \overset{\pounds.}{257} \,.\, \overset{s.}{18} \,.\, \overset{d.}{5\frac{1}{2}}.\tfrac{34}{61}, \text{ or } \overset{\pounds.}{17} \,.\, \overset{s.}{8} \,.\, \overset{d.}{2\frac{1}{4}}.\tfrac{27}{61}.$$

Also, since £106. 15*s*. pays £6. 15*s*. as discount for 18 months, we may evidently find the discount at once, as follows: thus,

$$\overset{\pounds.}{106} \,.\, \overset{s.}{15} : \overset{\pounds.}{275} \,.\, \overset{s.}{6} \,.\, \overset{d.}{8} :: \overset{\pounds.}{6} \,.\, \overset{s.}{15} : \overset{\pounds.}{x};$$

whence,

$$x = \frac{\overset{\pounds.}{275\frac{1}{3}} \times 6\frac{3}{4}}{106\frac{3}{4}} = \frac{\overset{\pounds.}{3304} \times 27}{1281 \times 4} = \frac{\overset{\pounds.}{1062}}{61} = \overset{\pounds.}{17} \,.\, \overset{s.}{8} \,.\, \overset{d.}{2\frac{1}{4}}.\tfrac{27}{61}$$

as before: and therefore the present worth is

$$\overset{\pounds.}{275} \,.\, \overset{s.}{6} \,.\, \overset{d.}{8} - \overset{\pounds.}{17} \,.\, \overset{s.}{8} \,.\, \overset{d.}{2\frac{1}{4}}.\tfrac{27}{61}, \text{ or } \overset{\pounds.}{257} \,.\, \overset{s.}{18} \,.\, \overset{d.}{5\frac{1}{2}}.\tfrac{34}{61}:$$

and from these steps we derive the following rules.

For the Present Worth. As the amount of £100. for the given time at the given rate : the proposed sum :: £100. : the present worth.

For the Discount. As the amount of £100. for the given time at the given rate : the proposed sum :: the interest of £100. for that time : the discount.

If no time be mentioned, the discount for a year is understood; and it may be observed that the statements above given are exactly like those of the *Rule of Three*, which will generally be worked in the manner there pointed out: and that the interest and amount of £100. are found by the *same* rule, or by *Practice*.

143. In the example above given, we see that

£.	s.		£.	s.	d.		£.	s.	
106	. 15	:	275	. 6	. 8	::	6	. 15	: the *discount;*

also, for 18 months or $1\frac{1}{2}$ year, we have

£.		£.	s.	d.		£.	s.	
100	:	275	. 6	. 8	::	6	. 15	: the *interest;*

whence, as the first term in the *former* proportion is greater than that in the *latter*, and the second and third terms are the same in both, it follows that the *interest* of a sum of money for any time is *greater* than its *discount* for the same time.

144. In the discharge of a Tradesman's bill, it is *customary* to deduct the *interest* at 5 per cent. for the given time, which is therefore to the payer's advantage: but Bankers, in *discounting* a bill or promissory note, are in the habit of charging *interest* at 5 per cent. from the day the bill is discounted, to the time when the *3 days of grace*, usually allowed, have elapsed; and this is of course an advantage to themselves, but still, perhaps, no greater than they are entitled to, in consequence of the hazard they run of the bill's not being *punctually* paid.

Examples for Practice.

(1) What is the present worth of £157. 10*s*. due 1 year hence, at 5 per cent.?

Answer: £150.

(2) Required the discount of £355. 5*s*. payable at the end of 4 months, at $4\frac{1}{2}$ per cent.

Answer: £5. 5*s*.

(3) Find the discount of £283. 0s. 5d. for 7 months, at 5 per cent.

Answer: £8. 0s. 5d.

(4) Determine the discount due upon £690. 3s. 9d. for 9 months, at 3 per cent.

Answer: £15. 3s. 9d.

(5) Find the discount of £298. 0s. 10d. for 11 months, at 4 per cent.

Answer: £10. 10s. 10d.

(6) Required the present worth of £370. 4s. $8\frac{1}{4}d.$ due 15 months hence, at $4\frac{5}{8}$ per cent.?

Answer: £350.

(7) What is the present worth of £325. 16s. 8d. due at the end of 5 months, at $4\frac{1}{2}$ per cent.?

Answer: £319. 16s. $1\frac{1}{2}d.$ $\frac{154}{163}$.

(8) Required the present worth of £241. 12s. 4d. due at the end of 146 days, at $4\frac{1}{3}$ per cent.; and shew that it will amount to this sum in the same time, at the same rate.

Answer: £237. 10.

IV. EQUATION OF PAYMENTS.

145. Def. The *Equation of Payments* is the finding of a proper time at which two or more debts due at *different* times should be discharged at *one* payment: and it is here *assumed* that the interests of *all* the debts for their respective periods, are together equal to the interest of their *sum* for the *Equated Time*.

Ex. If £100. be due in 3 months, £210. in 2 months and £160. in 5 months, find the equated time.

What is assumed in the definition leads immediately to the following equality, since the interests are proportional to the sums and times *jointly*, the rate being supposed the same;

$(100 \times 3) + (210 \times 2) + (160 \times 5) = (100 + 210 + 160) \times$ the equated time: whence, we have

$$\text{the equated time} = \frac{1520}{470} = 3\tfrac{11}{47} \text{ months}:$$

and thence the following rule is obtained.

Rule. Divide the sum of the products which arise from multiplying each payment by its time, by the sum of all the payments, and the quotient will be the equated time.

The assumption made in the definition, which implies that the *interest* of the debts payable *before* the equated time, from their times to the equated time, should be equal to the *interest* of the debts payable *after* that time, from the equated time to their respective times, is not founded in *equity;* because it is evident that by paying a debt before it is due, the debtor is entitled to the *discount* only, and that he virtually loses the *interest* which would have accrued from a debt, remaining in his hands after its period has expired.

We have seen in Article (143), that interest is greater than discount, and consequently the rule above laid down is in *favour* of the payer, since a greater allowance is made him than he is really entitled to: but as great nicety is not required, the equated time thus found will not be far from the truth: and indeed the *correct* time cannot easily be found without having recourse to *other* than arithmetical principles.

V. THE RULE OF FELLOWSHIP.

146. Def. *Fellowship* is the rule by means of which, two or more persons having a *Joint Stock,* or *Common Interest* in a property, are enabled to determine their respective shares of it, or of its profits, under the same or different circumstances.

Ex. 1. Two persons form a joint stock by subscribing £3500. and £5000. respectively, and in a certain time, they clear £1000.: how must this sum be divided between them?

Here, it is manifest that the share of each person must have the same ratio to the whole gain, that his subscription has to the whole stock formed; or, that the *whole* cause must be to each *partial* cause, as the *whole* effect is to each *partial* effect:

now, £3500 + £5000 = £8500 is the whole cause,

and £3500 and £5000 are the partial causes:

also, £1000 is the whole effect, and the partial effects are the required shares: whence, we have

$$£8500 : £3500 :: £1000 : \text{the first share};$$

$$\text{or, the first share} = \frac{3500 \times 1000}{8500} = \overset{£.}{411} . \overset{s.}{15} . \overset{d.}{3\tfrac{1}{2}} . \tfrac{2}{17}:$$

$$£8500 : £5000 :: £1000 : \text{the second share};$$

$$\text{or, the second share} = \frac{5000 \times 1000}{8500} = \overset{£.}{588} . \overset{s.}{4} . \overset{d.}{8\tfrac{1}{4}} . \tfrac{15}{17}:$$

and these sums evidently make up the whole £1000 gained.

Here, the ratio of the shares depending *solely* upon the amounts respectively subscribed, the example is termed an instance of *Single* Fellowship.

Ex. 2. A field of grass is rented by two persons for £27.: the former keeps in it 15 oxen for 10 days, and the latter 21 oxen for 7 days; find the rent paid by each of them, on the supposition that the pasturage remains equally good throughout.

Here, the portions of the rent must evidently be as the numbers of oxen and the numbers of days *jointly*: also, the partial causes are

$$15 \times 10 = 150, \text{ and } 21 \times 7 = 147:$$

and therefore the whole cause is 150 + 147 or 297; whence as before, we have

$$297 : 150 :: 27 : \overset{£.}{13} . \overset{s.}{12} . \overset{d.}{8\tfrac{1}{2}} . \tfrac{10}{11}, \text{ the 1st portion}:$$

$$297 : 147 :: 27 : 13 . \ 7 . 3\tfrac{1}{4} . \tfrac{1}{11}, \text{ the 2nd portion}:$$

and the sum of both portions is £27., as it ought.

This is an instance of *Double* Fellowship, the portions of rent depending upon *two* particulars: the number of oxen put in, and the number of days they are kept there.

147. The principles of these examples, being independent of the *number* of interests concerned, enable us to lay down the following rule.

Rule. Find the relative values of the partial causes, and also their sum: then, as this sum is to each part of it, so is the whole effect to its corresponding part.

In this rule it is understood that every agent is employed under exactly the same circumstances: as, for instance, in the last example, each of the oxen is supposed to consume the same quantity of grass, the pasturage being uniform throughout: but whenever their *relative* qualities are assigned, it will easily be seen that the same method must be pursued.

Ex. If £100. be distributed among 6 men, 9 women and 12 children; what will they receive, when the shares of a man, woman and child, are as the numbers 3, 2, 1?

$$\text{Here,}\quad \left.\begin{aligned} 6 \times 3 &= 18 \\ 9 \times 2 &= 18 \\ 12 \times 1 &= 12 \end{aligned}\right\} \text{ are the partial causes:}$$

$$\text{and}\quad \overline{\underline{\;48\;}}\quad \text{is the whole cause:}$$

whence, $48 : 18 :: 100 : 37 \,.\, 10$, to the men:
$48 : 18 :: 100 : 37 \,.\, 10$, to the women:
$48 : 12 :: 100 : 25 \,.\, 0$, to the children.

VI. THE RULE OF ALLIGATION.

148. DEF. *Alligation*, sometimes called *Alligation Medial*, is the rule by means of which, the rate or quality of a composition or mixture, is determined from the rates or qualities of the ingredients of which it is made up.

Ex. If 12 bushels of wheat at 6*s*. a bushel, and 15 bushels at 7*s*. a bushel, be mixed together; what will be the value of a bushel of the mixture?

Here, from the most obvious principles, we have

$$\left.\begin{aligned} 12 \times 6 &= 72 \\ 15 \times 7 &= 105 \end{aligned}\right\} \text{ the values of the ingredients:}$$

therefore $72 + 105 = 177s.$ the value of the mixture, which contains $12 + 15 = 27$ bushels: whence,

$$\overset{\text{bush.}}{27} : \overset{\text{bush.}}{1} :: \overset{s.}{177} : \overset{s.}{6} \,.\, \overset{d.}{6\tfrac{2}{3}}.\tfrac{2}{3}, \text{ the price of a bushel.}$$

The usual form of the operation is as follows:

$$\begin{aligned} 6 \times 12 &= 72 \\ 7 \times 15 &= 105 \end{aligned}$$

$$27)\ 177\ (\overset{s.}{6} \cdot \overset{d.}{6\tfrac{1}{2}} \cdot \tfrac{2}{3}:$$

and the number of ingredients being any whatever, we shall have the following rule.

RULE. Divide the sum of the products of the ingredients and their respective rates, by the sum of the ingredients; and the quotient will be the rate of the mixture.

Examples for Practice.

(1) If £75. be due in 4 months, £125. in 5 months and £150. in 7 months: what is the equated time of payment?

Answer: $5\frac{9}{14}$ months.

(2) What will be the equated time of payment of £150. 17*s*. 6*d*. due at 4 months, £175. 16*s*. 8*d*. at 6 months, and £325. 18*s*. 9*d*. at 8 months?

Answer: $6\frac{16806}{31887}$ months.

(3) Find the equated time of payment, when $\frac{1}{2}$ of a sum of money is due at 3 months, $\frac{1}{5}$ at 8 months, and the remainder at 15 months.

Answer: $7\frac{3}{5}$ months.

(4) Divide £1000. among three persons, so that their shares shall be as the numbers 2, 5, and 9.

Answer: £125., £312. 10*s*., and £562. 10*s*.

(5) Three partners put into business the sums of £3000., £5250. and £6825.; and at the end of a certain time they gain £2000.: find the share of each?

Answer: £398. 0*s*. $2\frac{1}{4}$*d*. $\frac{37}{67}$, £696. 10*s*. 4*d*. $\frac{48}{67}$, and £905. 9*s*. $5\frac{1}{4}$*d*. $\frac{49}{67}$.

(6) Three merchants *A, B, C*, engage in commerce; *A* with £1000. for 12 months, *B* with £1800. for 7 months, and *C* with £2500. for 4 months; and they gain £350: what share of the profit belongs to each?

Answer: £121. 7*s*. $8\frac{3}{4}$*d*. $\frac{137}{173}$ to *A*, £127. 9*s*. $1\frac{1}{2}$*d*. $\frac{66}{173}$ to *B*, and £101. 3*s*. $1\frac{1}{4}$*d*. $\frac{143}{173}$ to *C*.

(7) A wine merchant mixes together 20 gallons of wine at 12*s.* a gallon; 25 gallons at 14*s.* and 36 gallons at 16*s.*: what will be the price of a gallon of the mixture?

Answer: 14*s.* $4\frac{1}{2}d.\ \frac{26}{27}$.

(8) A mixture is made of 10 bushels of flour at 3*s.* 8*d.*, 21 bushels at 3*s.* 10*d.*, and 35 bushels at 4*s.*: what is the price of a bushel of it?

Answer: 3*s.* $10\frac{3}{4}d.\ \frac{1}{33}$.

VII. THE DOCTRINE OF EXCHANGES.

149. DEF. 1. *Exchange* is the rule by means of which it is ascertained what sum of money of one country is equivalent to any *given* sum of another, according to some *settled* rate of commutation: and it is evident that the operations necessary to effect this, must, from the nature of the case, be merely applications of the Rule of Proportion.

The *Course of Exchange* is used to express the sum of money of any place given in exchange for a *fixed* sum of that of another: and the *Par of Exchange* denotes the sum of money of any place, which is of the same *intrinsic* value as that fixed sum.

Ex. How many pounds Flemish can I receive for £1050. sterling, the course of exchange being 35 shillings Flemish for £1. sterling?

Here, from the nature of the question, we have

```
      £.     £.     s.f.
      1 : 1050 :: 35
               35
          -------
            5250
           3150
          -------
  2,0 )3675,0 shillings Flemish:
      ----------
      £ 1837 . 10 the sum Flemish required;
      ----------
```

and it may be remarked that, in questions of this nature, all that is necessary to be known is the course of exchange, and the subdivisions of the monies to be commuted.

150. DEF. 2. The *Arbitration*, or *Comparison* of Exchanges, is the determining what rate of exchange called the *Par of Arbitration*, between any number of places corresponds with, or is equivalent to, any assigned rates between each of them and another place: and a competent knowledge of this subject will, of course, enable a person to judge how he may remit his money from one place to another, with the greatest possible advantage.

Arbitration is styled *simple* or *compound*, according as *three* or *more* places are concerned.

Ex. If the exchange between *Amsterdam* and *Paris* be 54*d.* for 1 crown, and between *Amsterdam* and *London* 33*s.* 9*d.* for £1.; what is the par of exchange, or the arbitrated price between *Paris* and *London*?

Here, 1 crown at *Paris* = 54 pence at *Amsterdam*:

240 pence in *London* = 405 pence at *Amsterdam*:

thus, we obtain the equality of ratios,

$$\frac{\text{1 crown at } Paris}{\text{240 pence in } London} = \frac{54}{405} = \frac{2}{15}:$$

$$\text{whence, 1 crown at } Paris = \frac{2}{15} \times \overset{d.}{240} = \overset{d.}{32} \text{ in } London:$$

that is, 32*d.* per crown is the arbitrated price between *London* and *Paris*.

If we arrange the equalities, so that the first term of one shall always be of the *same kind* as the second of that which immediately precedes it, as follows:

1 crown at *Paris* = 54 pence at *Amsterdam*;

405 pence at *Amsterdam* = 240 pence in *London*,

and multiply together the corresponding terms, retaining the *names* only of the first and last countries and their *denominations* of money, we shall have

405 crowns at *Paris* = 54 × 240 pence in *London*;

$$\text{and therefore 1 crown at } Paris = \frac{54 \times \overset{d.}{240}}{405} = \overset{d.}{32} \text{ in } London,$$

as before: and a proceeding of this kind is sometimes distinguished by the name of the *Chain Rule*, from the

connection of the *first* and *last* terms, thus ascertained through those which are *intermediate*.

The reader, who may be desirous of extending his knowledge upon this subject, is referred to the last Edition of Dr Kelly's *Universal Cambist*.

VIII. MISCELLANEOUS QUESTIONS.

151. In this section are presented a few miscellaneous Questions, which could not with propriety be arranged under any of the preceding heads, and are still of too much importance to be passed over without notice, in a work like the present.

Qu. 1. How many dozens of wine at £2. a dozen, must be given in exchange for 27 yards of broad cloth at 32*s*. a yard?

The price of the cloth is $27 \times 32 = 864s.$:

whence, $40s. : 864s. :: 1\,\text{doz.} : 21\tfrac{3}{5}\,\text{doz.}$;

that is, $21\frac{3}{5}$ dozens of wine are of equal value with 27 yards of cloth.

Questions of this kind are sometimes termed instances of *Barter* or *Truck*.

Qu. 2. If a grocer by selling tea at 6*s*. 6*d*. a pound, clear one-sixth of the money: what will he clear per cent. by selling it at 7*s*. a pound?

$$\text{Here, } \frac{1}{6} \text{ of } 6s.\ 6d. = 1s.\ 1d.;$$

and therefore 5*s*. 5*d*. a pound is the price the tea cost him: whence,

$$5s.\ 5d. : 7s. :: £100. : £129.\ 4s.\ 7\tfrac{1}{4}d.\tfrac{7}{18};$$

and therefore, £129. 4*s*. $7\frac{3}{4}d.\frac{7}{13}$, is the *increased* value of £100. at this rate: that is, the *gain* per cent. is

$$£29.\ 4s.\ 7\tfrac{1}{4}d.\tfrac{7}{18}.$$

Questions of this description are generally classed under the heads, *Profit and Loss*, or *Loss and Gain*.

Qu. 3. Required the neat weight of 27 cwt. 1 qr. 14 lbs., tare being allowed at the rate of 16 lbs. per cwt.

Here, by the rules of Practice before given, we have

$$\begin{array}{r|c|l}
16 & \frac{1}{7} & \overset{\text{cwt.}}{27} \;.\; \overset{\text{qr.}}{1} \;.\; \overset{\text{lbs.}}{14} = \text{gross:} \\
 & & \;\;3 \;.\; 3 \;.\; 18 = \text{tare:} \\
\hline
 & & 23 \;.\; 1 \;.\; 24 = \text{neat weight.}
\end{array}$$

Questions of this nature are usually inserted under a rule called *Tare and Tret*, which comprises all allowances made upon goods, on any ground whatever, whether by *custom* or by *special agreement*.

Qu. 4. If two men A and B together can finish a piece of work in 10 days, and A by himself can do it in 18 days: what time will it take B to do the whole?

Assuming 1 to represent the piece of work, we have

$$\frac{1}{18} = \text{work done by } A \text{ in } 1 \text{ day:}$$

$$\frac{10}{18} = \frac{5}{9} = \ldots\ldots\ldots\ldots\ldots \; A \text{ in } 10 \text{ days:}$$

$$\text{hence, } 1 - \frac{5}{9} = \frac{4}{9} = \ldots\ldots\ldots\ldots\ldots \; B \text{ in } 10 \text{ days:}$$

$$\text{wherefore, } \frac{4}{9} : 1 :: 10 \text{days} : 22\tfrac{1}{2}\text{days;}$$

or, B can do the whole work in $22\frac{1}{2}$ days.

Qu. 5. Three agents A, B, C, can produce a given effect in 12 hours; also, A and B can produce it in 16 hours, and A and C in 18 hours: in what time can each of them produce it separately?

Here, reasoning as before, we shall have

$$\frac{1}{16} = \text{effect produced by } A \text{ and } B \text{ in } 1 \text{ hour:}$$

$$\frac{12}{16} = \frac{3}{4} = \ldots\ldots\ldots\ldots\ldots\ldots \; A \text{ and } B \text{ in } 12 \text{ hours:}$$

$$\text{whence, } 1 - \frac{3}{4} = \frac{1}{4} = \ldots\ldots\ldots\ldots\ldots\ldots \; C \text{ in } 12 \text{ hours:}$$

$$\text{and therefore, } \frac{1}{4} : 1 :: 12\text{hrs.} : 48\text{hrs.,}$$

the time in which C alone can produce it:

$$\text{again, } \frac{1}{18} = \text{effect produced by } A \text{ and } C \text{ in } 1 \text{ hour:}$$

$$\frac{12}{18} = \frac{2}{3} = \ldots\ldots\ldots\ldots\ldots\ldots \; A \text{ and } C \text{ in } 12 \text{ hours:}$$

$$\text{and } 1 - \frac{2}{3} = \frac{1}{3} = \ldots\ldots\ldots\ldots\ldots\ldots \; B \text{ in } 12 \text{ hours:}$$

$$\text{whence, } \frac{1}{3} : 1 :: 12\text{hrs.} : 36\text{hrs.},$$

the time in which B alone can produce it:

$$\text{also, } \frac{1}{12} = \text{effect produced by } A, B \,\&\, C \text{ in } 1 \text{ hour:}$$

$$\text{and } \frac{1}{36} + \frac{1}{48} = \frac{7}{144} = \ldots\ldots\ldots\ldots\ldots \; B \text{ and } C \text{ in } 1 \text{ hour:}$$

$$\text{whence, } \frac{1}{12} - \frac{7}{144} = \frac{5}{144} = \ldots\ldots\ldots\ldots \; A \text{ in } 1 \text{ hour:}$$

$$\text{therefore, } \frac{5}{144} : 1 :: 1\text{hr.} : 28\tfrac{4}{5}\text{hrs.},$$

the time in which A can produce the effect proposed.

Qu. 6. Distribute £200. among A, B, C and D, so that B may receive as much as A; C as much as A and B together; and D as much as A, B and C together.

If the share of A be represented by 1; then will
the share of B be represented by 1:
the share of C by $1 + 1 = 2$:
and the share of D by $1 + 1 + 2 = 4$:

whence, the question is merely to divide £200. into four parts having the same proportions as the numbers 1, 1, 2, 4:

$$\text{also, } 1 + 1 + 2 + 4 = 8,$$

and the Rule of Fellowship gives the following proportions:

		£.	£.	
8 : 1	::	200 :	25,	the share of A;
8 : 1	::	200 :	25,	the share of B;
8 : 2	::	200 :	50,	the share of C;
8 : 4	::	200 :	100,	the share of D.

The same mode of reasoning will be applicable, whatever be the number of persons concerned.

Qu. 7. At what times between 2 and 3 o'clock, are the hour and minute hands of a clock together; at right angles; and in opposite directions?

At two o'clock, the hour hand is *two* of the portions, called *hours* of one hand and *five minutes* of the other, in advance of the minute hand; and their rates being as 1 : 12, the minute hand *gains* 55 in 60, or 11 in 12, upon the hour hand: whence we have

$$11 : 12 :: 2 : 2\tfrac{2}{11},$$

the time at which the minute and hour hand are together.

Again, when they are at right angles, the minute hand must have gained $2 + 3 = 5$ portions; and we have

$$11 : 12 :: 5 : 5\tfrac{5}{11};$$

and therefore at $5\frac{5}{11} \times 5$ or $27\frac{3}{11}$ minutes past two, the hands are at right angles.

Also, if they point in opposite directions, $2 + 6 = 8$ portions must be gained by the minute hand; and therefore we have

$$11 : 12 :: 8 : 8\tfrac{8}{11},$$

or, the hands will be in opposite directions at $8\frac{8}{11} \times 5$, or $43\frac{7}{11}$ minutes past two.

When the minute hand has gained $2 + 9 = 11$ portions, the two hands will be at right angles again; and

$$11 : 12 :: 11 : 12,$$

which shews that this circumstance occurs at 60 minutes past two, or at three o'clock, as we know to be the case.

Qu. 8. Two clocks point out 12 at the same instant: one of them gains 7″ and the other loses 8″ in 12 hours: after what interval will one have gained half an hour of the other, and what o'clock will each then shew?

Here, $7'' + 8'' = 15''$, is the separation which takes place in 12 hours; and $\frac{1}{2}$ hour $= 30' = 1800''$: whence,

$$15'' : 1800'' :: 12\text{hrs.} : 1440\text{hrs.};$$

that is, in 1440 hours or 60 days, they will be separated 30 minutes or half an hour.

Also, the first gains 7″ in 12 hours, or 14″ in 1 day:

$$\text{and } 1\text{day} : 60\text{days} :: 14'' : 14',$$

and therefore it will shew 12 hours 14 minutes.

The second loses $8''$ in 12 hours, or $16''$ in 1 day; and

$$1\text{ day} : 60\text{ days} :: 16'' : 16';$$

whence the time pointed out by it will be 12h. − 16′, or 11 hours 44 minutes: and it will be observed that these times differ by half an hour, as they ought.

Examples for Practice.

(1) How much cloth at 14*s.* 6*d.* a yard, must be given for 3 cwt. 3 qrs. of sugar, at £3. 4*s.* per cwt.?

Answer: 16 yds. $2\frac{8}{29}$ qrs.

(2) If 126 yards of cloth be bartered for 3 hhds. of brandy, at 6*s.* 8*d.* per gallon: what is the price of the cloth per yard?

Answer: 10*s.*

(3) If I buy goods at £3. 16*s.* 8*d.* per cwt.: how must I retail them per lb., to gain 15 per cent.?

Answer: $9\frac{1}{4}d.\frac{11}{14}$.

(4) If by selling tea at 6*s.* 4*d.* per lb., a grocer lose 6 per cent.: what did it cost him per lb.?

Answer: 6*s.* $8\frac{3}{4}d.\frac{19}{47}$.

(5) A grocer bought 2 tons. 3 cwt. 3 qrs. of sugar for £120., and paid £2. 10*s.* for expences: what must he sell it at per cwt. to clear 50 per cent.?

Answer: £4. 4*s.*

(6) A person, by disposing of goods for £182., loses at the rate of 9 per cent.: what ought they to have been sold for, to realize a profit of 7 per cent.?

Answer: £214.

(7) Bought 2688 yards of cambric at 8*s.* 8*d.* a yard, and sold $\frac{1}{4}$ at 10*s.* 2*d.*; $\frac{1}{3}$ at 10*s.* $11\frac{1}{3}d.$, and the remainder at 11*s.* $4\frac{1}{2}d.$ a yard: what is the whole gain, and also the gain per cent.?

Answer: £304. 14*s.* 8*d.*, and £26. 3*s.* $2\frac{3}{4}d.\frac{5}{13}$.

(8) A stationer sold quills at 11*s.* a thousand, by which he cleared $\frac{3}{8}$ of the money, and he afterwards raised them to 13*s.* 6*d.* a thousand: what did he clear per cent. by the latter price?

Answer: £96. 7*s.* $3\frac{1}{4}d.\frac{1}{11}$.

(9) At what price must a commodity, purchased at the rate of £14. 5*s.* per cwt., be sold to gain 21 per cent.; and what quantity of it must be sold at that rate to clear £100.?

Answer: at £17. 4*s.* $10\frac{1}{5}d.$ per cwt., and the quantity of it, is 33cwt. 1qr. 18lbs. $11\frac{7}{171}$oz.

(10) A merchant bought 160 quarters of wheat at 41*s.* 3*d.* per quarter, and sold it at 58*s.* 4*d.*: what was his gain? At what price ought it to have been sold to gain exactly £100?

Answers: £136. 13*s.* 4*d.*, and 53*s.* 9*d.*

(11) If a parcel of goods bought for £18., be sold four months afterwards for £25.; what is the gain per cent. per annum?

Answer: £116. 13*s.* 4*d.*

(12) Divide £64. among *A*, *B* and *C*, so that *A* may have three times as much as *B*; and *C* may have one third of what *A* and *B* have together.

Answer: *A* has £36., *B* has £12., and *C* has £16.

(13) A person paid a tax of 10 per cent. upon his income: what must his income have been, when after he had paid the tax, there was £1250. remaining?

Answer: £1388. 17*s.* $9\frac{1}{4}d.\frac{1}{3}$.

(14) A grocer had 150lbs. of tea, of which he sold 50lbs. at 9*s.* per lb., and found that he was thereby gaining $7\frac{1}{2}$ per cent.; at what rate must he sell the remaining 100lbs., so as to clear 10 per cent. upon the whole?

Answer: 9*s.* $3\frac{3}{4}d.\frac{8}{43}$.

(15) A mixture of wine and water of 32 measures contains one measure of wine: how much water must be added to this mixture, that 32 measures of it may contain $\frac{1}{8}$ of a measure of wine?

Answer: 224 measures.

(16) A hare starts 40 yards before a greyhound, and is not perceived by him till she has been up 40 seconds: she scuds away at the rate of 10 miles an hour, and the dog pursues her at the rate of 18 miles an hour: how long will the course last, and what distance will the hare have run?

Answer: $60\frac{5}{22}$ seconds, and 490 yards.

(17) At what time, between twelve and one o'clock, do the hour and minute hands of a watch point in directions exactly opposite?

Answer: 32 min. $43\frac{7}{11}$ sec. past 12.

(18) If 5 men or 7 women can perform a piece of work in 35 days: in what time can 7 men and 5 women do the same?

Answer: $16\frac{41}{74}$ days.

(19) If 15 men, 12 women, and 9 boys can complete a piece of work in 50 days; what time would 9 men, 15 women, and 18 boys take to do twice as much, the parts done by each in the same time being as the numbers 3, 2 and 1?

Answer: 104 days.

(20) If *A* by himself can do a piece of work in 5 days; *B* twice as much in 7 days, and *C* four times as much in 11 days: in what time can *A*, *B* and *C* together do three times the said work?

Answer: 3 days. 12 hrs. $46\frac{26}{109}$ min.

(21) If *A* and *B* together can build a boat in 18 days, and with the assistance of *C* they can do it in 11 days; in what time can *C* do it by himself?

Answer: $28\frac{2}{7}$ days.

(22) If *A* can do a piece of work by himself in 1 hour, *B* in 3 hours, *C* in 5 hours, and *D* in 7 hours: in what time can they do three times as much, all working together?

Answer: 1 hour. 47 min. $23\frac{2}{11}$ sec.

(23) *A* and *B* can do a piece of work in 10 days; *A* and *C* in 12 days, and *B* and *C* in 14 days: in what times can they do it jointly and separately?

Answer: All together in $7\frac{91}{107}$ days; *A* in $17\frac{41}{47}$ days; *B* in $22\frac{26}{37}$ days, and *C* in $36\frac{12}{23}$ days.

(24) If *A*, *B* and *C* could reap a field in 18 days; *B*, *C* and *D* in 20 days; *C*, *D* and *A* in 24 days, and *D*, *A* and *B* in 27 days: in what times would it be reaped by them all together, and by each of them separately?

Answer: By them altogether in $16\frac{56}{199}$ days: by *A* in $87\frac{21}{37}$ days: by *B* in $50\frac{5}{8}$ days: by *C* in $41\frac{1}{79}$ days, and by *D* in $170\frac{10}{19}$ days.

CHAPTER VII.

INVOLUTION AND EVOLUTION,

WITH THE ARITHMETIC OF SURDS.

INVOLUTION.

152. Def. A *Power* of any number or quantity is the number or quantity which arises from successive multiplications by itself: the operation by which it is obtained is termed *Involution*; and the *Degree* or *Order* of the power is denoted by the *number* of equal factors employed.

Thus, taking the number 2, we shall have the following powers of it:

$2 = 2$, the first power of 2:

$2 \times 2 = 4$, the second power of 2:

$2 \times 2 \times 2 = 8$, the third power of 2:

$2 \times 2 \times 2 \times 2 = 16$, the fourth power of 2:

$2 \times 2 \times 2 \times 2 \times 2 = 32$, the fifth power of 2:

$2 \times 2 \times 2 \times 2 \times 2 \times 2 = 64$, the sixth power of 2:

and so on, as far as we please:

but instead of expressing these multiplications at length, which would soon become inconvenient, we denote the same operations by means of *Indices*, or small figures placed a little above the line to the right of the quantities whose powers are intended to be exhibited: thus, what is given above may be denoted by,

$$2^1 = 2: \quad 2^3 = 8: \quad 2^5 = 32:$$
$$2^2 = 4: \quad 2^4 = 16: \quad 2^6 = 64: \text{ \&c}:$$

where it is evident that the *Index*, sometimes called the *Exponent*, is *equal* to the number of *factors* employed, and greater by *one* than the number of *operations*.

In the same manner, the *second* powers of the first nine digits are expressed: thus,

$$1^2 = 1: \qquad 4^2 = 16: \qquad 7^2 = 49:$$
$$2^2 = 4: \qquad 5^2 = 25: \qquad 8^2 = 64:$$
$$3^2 = 9: \qquad 6^2 = 36: \qquad 9^2 = 81:$$

and their *third* powers will be as follows:

$$1^3 = 1: \qquad 4^3 = 64: \qquad 7^3 = 343:$$
$$2^3 = 8: \qquad 5^3 = 125: \qquad 8^3 = 512:$$
$$3^3 = 27: \qquad 6^3 = 216: \qquad 9^3 = 729.$$

The second and third powers of numbers are generally styled their *Squares* and *Cubes*, in reference to their application to *Geometry*, as will be seen hereafter: and the operations by which all powers are obtained, are merely those of Multiplication.

153. To find the powers of a vulgar fraction, or of a quantity expressed decimally, a similar process is used: thus,

$$\left(\frac{2}{3}\right)^1 = \frac{2}{3}: \qquad (2.5)^1 = 2.5:$$

$$\left(\frac{2}{3}\right)^2 = \frac{2}{3} \times \frac{2}{3} = \frac{4}{9}: \qquad (2.5)^2 = 2.5 \times 2.5 = 6.25:$$

$$\left(\frac{2}{3}\right)^3 = \frac{2}{3} \times \frac{2}{3} \times \frac{2}{3} = \frac{8}{27}: \qquad (2.5)^3 = 2.5 \times 2.5 \times 2.5 = 15.625:$$

&c.: &c.:

and exactly in the same manner, the powers of a quantity expressed by factors are found:

$$\text{thus, the square of } 2 \times 7 = (2 \times 7) \times (2 \times 7)$$
$$= 2 \times 2 \times 7 \times 7 = 2^2 \times 7^2 = 4 \times 49 = 196.$$

Hence it appears that any power of a fraction is equal to the fraction formed by *raising* both its numerator and denominator to that power: and that the power of a quantity formed by factors is found by raising each factor to that power. A mixed fractional quantity is generally represented as a simple fraction, or as a decimal, before the process is applied.

154. This method of notation furnishes some important conclusions with respect to powers generally.

Thus, since $3^2 = 3 \times 3$, and $3^4 = 3 \times 3 \times 3 \times 3$: we have

$$3^4 \times 3^2 = (3 \times 3 \times 3 \times 3) \times (3 \times 3)$$
$$= 3 \times 3 \times 3 \times 3 \times 3 \times 3 = 3^6 = 3^{4+2}:$$
$$3^4 \div 3^2 = (3 \times 3 \times 3 \times 3) \div (3 \times 3)$$
$$= \frac{3 \times 3 \times 3 \times 3}{3 \times 3} = 3 \times 3 = 3^2 = 3^{4-2}:$$

from which we infer that the *Multiplication* and *Division* of different powers of the same quantity, are expressed by the *Addition* and *Subtraction* of their indices.

Similarly, we have the fourth power of 3^2, expressed by $3^2 \times 3^2 \times 3^2 \times 3^2 = 3^8 = 3^{2\times 4}$; or, the *Involution* of powers is expressed by the *Multiplication* of their indices, and conversely.

Ex. Let it be required to find the 6th power of 13.

$$\text{Here, } 13^6 = 13^{1\times 2\times 3} = 13^1 \times 13^2 \times 13^3$$
$$= 13 \times 169 \times 2197 = 4826809:$$

and the same result will evidently be obtained by effecting any of the operations indicated below:

$$13^6 = 13^2 \times 13^4 = 13^3 \times 13^3 = 13^5 \times 13.$$

155. When one power of a quantity is divided by a higher power of the same quantity, the quotient may be expressed by the power of a fraction: thus,

$$7^2 \div 7^4 = (7 \times 7) \div (7 \times 7 \times 7 \times 7)$$
$$= \frac{7 \times 7}{7 \times 7 \times 7 \times 7} = \frac{1}{7 \times 7} = \frac{1}{7^2} = \left(\frac{1}{7}\right)^2.$$

Also, from these articles we ascertain that,

$$7^4 \div 7^2 = 7^{4-2} = 7^2:$$
$$7^2 \div 7^4 = \frac{1}{7^{4-2}} = \frac{1}{7^2}:$$

where the *difference* of the indices is employed in the numerator or denominator, according as the dividend or divisor is the higher power.

If the indices of the dividend and divisor be the same, this notation *extended* will give us the representation of unity or 1, in the *form* of the power of any number or quantity whatever, as 7 for instance, whose index is 0, since,

$$1 = 7^4 \div 7^4 = 7^{4-4} = 7^0. \quad \therefore 7^0 = 1.$$

EVOLUTION.

156. Def. A *Root* of a number or quantity is such a number or quantity as being multiplied into itself one or more times produces it; and the operation by which this root is obtained is called *Evolution*.

Thus, the second or *square* root of 16 is 4, because the square of 4, or $4^2 = 4 \times 4 = 16$.

The third or *cube* root of 512 is 8, since the cube of 8,

$$\text{or } 8^3 = 8 \times 8 \times 8 = 512:$$

and similarly of vulgar fractions and decimals.

This operation is generally expressed by means of the sign $\sqrt{\ }$, which is called the *Radical Sign*, with a small figure placed on its left to *particularize* the root required: thus, the instances above given may be written,

$$\sqrt[2]{16} = 4, \text{ and } \sqrt[3]{512} = 8:$$

but the square root is frequently denoted by the sign $\sqrt{\ }$ only, without the small figure, as being of most frequent occurrence.

The same operations are also indicated by means of the primitive fractions $\frac{1}{2}$, $\frac{1}{3}$, &c., used as *indices:* so that the indices $\frac{1}{2}$, $\frac{1}{3}$, &c., denote operations exactly the reverse of those expressed by the indices 2, 3, &c., respectively: thus,

$$4^2 = 16, \quad 16^{\frac{1}{2}} = 4:$$
$$8^3 = 512, \quad 512^{\frac{1}{3}} = 8.$$

EXTRACTION OF THE SQUARE ROOT.

157. In this operation, having only one magnitude to work with, we shall not be entitled to avail ourselves directly of any of the fundamental operations of arithmetic: and we shall therefore merely put down such instructions as will enable the student to extract the square

root, without entering very particularly into the reasons upon which they are founded, these reasons admitting of a much clearer exposition by means of algebraical symbols, than any that could be given in particular numbers.

158. Repeating what was said in article (152), we have

Digits:

1, 2, 3, 4, 5, 6, 7, 8, 9:

Squares:

1, 4, 9, 16, 25, 36, 49, 64, 81:

whence, by mere inspection, we are enabled to find the square roots of all quantities that can be produced by the *squaring* of a single figure: but it is evident that this statement will not be sufficient for finding the square roots of quantities consisting of more than *two* figures, and recourse must therefore be had to other expedients.

159. *From the number of figures in any proposed quantity, to find the number of figures in its square root.*

Since, the square root of 1 is 1:
the square root of 100 is 10:
the square root of 10000 is 100:
the square root of 1000000 is 1000: &c.,

we see immediately that the square root of a number of fewer than three figures must consist of only one figure: that of a number of more than two figures and fewer than five, of two figures: that of a number of more than four figures and fewer than seven, of three figures, and so on: whence it follows, that if a dot or full point be placed over every alternate figure, beginning at the *units'* place, the number of such points will be the same as the number of figures in the square root.

This is called the *Rule for Pointing*, and may easily be extended to decimals: thus,

since, the square root of .01 is .1:
the square root of .0001 is .01:
the square root of .000001 is .001: &c.,

we infer that the quantity proposed must first be made to have an *even* number of decimal places, and then the pointing must proceed from the place of *units* towards

the right hand over every alternate figure as before: and the number of such points will be the same as the number of decimal places in the square root.

Rule for the Extraction of the Square Root.

Point the alternate figures of the number proposed, beginning at the place of units, so as to form as many *periods* of two figures each as possible: find the greatest square number contained in the first period on the left hand, put down its root on the right as in division, and subtract it from that period. To the remainder bring down the next period for a dividend, double the root just found for a divisor, and find how often it is contained in this dividend exclusive of the figure on its right hand, annex this quotient to the figures in both the quotient and divisor: multiply the divisor thus formed by the last figure of the quotient, subtract the product as before, and bring down to the remainder the period which comes next in order: repeat the process till every period in succession is disposed of, and the root, or an approximation to it, will thus be obtained.

The divisors *tried* as above will frequently be taken too large, when the dividend consists of only two or three figures, but not so in other cases: and attention to this circumstance will save trouble.

Ex. 1. Find the square roots of 1444 and 16129.

Proceeding according to the directions given in the rule, we have

$$\begin{array}{r|l} 1\dot{4}4\dot{4}\,(38 \\ 9 \\ \hline 68)\,544 \\ 544 \\ \hline \end{array}$$

$$\begin{array}{r} \dot{1}6\dot{1}2\dot{9}\,(127 \\ 1 \\ \hline 22)\,61 \\ 44 \\ \hline 247)\,1729 \\ 1729 \\ \hline \end{array}$$

or, the square roots of 1444 and 16129, are 38 and 127, respectively: and these operations may easily be verified by squaring the numbers 38 and 127: also, the importance of the remark above made will be apparent.

Ex. 2. Required the square roots of the mixed decimals, 22.09 and 104.7931.

$$\begin{array}{rl}
 & 2\dot{2}.0\dot{9}\,(4.7 \\
 & 16 \\ \hline
87) & 609 \\
 & 609 \\ \hline
\end{array}$$

$$\begin{array}{rl}
 & \dot{1}0\dot{4}.7\dot{9}3\dot{1}\,(10.23 \\
 & 1 \\ \hline
202) & 0479 \\
 & 404 \\ \hline
2043) & 7531 \\
 & 6129 \\ \hline
 & 1402 \\ \hline
\end{array}$$

The former of these is a complete square whose root is 4.7; but the latter is not, its approximate root being 10.23 with a remainder .1402: and it will be found upon trial, that $(10.23)^2 + .1402 = 104.7931$: also, this approximation might evidently have been carried farther, by affixing to the decimals of the quantity proposed, periods of *ciphers* which do not affect its value.

Ex. 3. Determine the square roots of the fractional quantities $\frac{144}{169}$ and $1278\frac{7}{25}$.

From article (153), we see that the square root of a fraction may be obtained by finding the square roots of its numerator and denominator separately: whence, the square root of $\frac{144}{169}$ will be found to be $\frac{12}{13}$.

Hence also, since $1278\frac{7}{25} = \frac{31957}{25}$, the square root may be found as above: but as the numerators and denominators are seldom complete squares, it is usual to express the fraction decimally before the rule is applied; and in this instance, we shall have the approximate square root of $1278.28 = 35.753$ &c., which might have been extended to more decimal places at pleasure.

Ex. 4. Extract the square root of the recurring decimal $1.\dot{7}$.

Here $1.\dot{7} = \frac{16}{9}$; and therefore the square root is $\frac{4}{3} = 1.\dot{3}$: but it generally happens that the corresponding

vulgar fraction is not a complete square, and the approximate root must then be found by the ordinary method, though it will not be a recurring decimal.

It may here be observed, that the remainder at any stage of the operation, must not exceed twice the corresponding quotient or portion of the root: and when a few figures of the root are obtained, their number may nearly be *doubled* by Division only.

Examples for Practice.

(1) Find the square roots of 676, 21025, 288369 and 998001.

Answers: 26, 145, 537 and 999.

(2) Determine the square roots of 2025, 692224, 33016516 and 45859984.

Answers: 45, 832, 5746 and 6772.

(3) What are the square roots of 5774409, 62805625, 182493081 and 3915380329?

Answers: 2403, 7925, 13509 and 62573.

(4) Required the square roots of 33.64, 1082.41, 22.8484 and 187.4161.

Answers: 5.8, 32.9, 4.78 and 13.69.

(5) Find the square roots of .0064, .005329, .00674041 and .00038025.

Answers: .08, .073, .0821 and .0195.

(6) Extract the square roots of $\frac{4}{25}$, $\frac{169}{256}$, $\frac{841}{1369}$ and $\frac{2304}{3481}$.

Answers: $\frac{2}{5}$, $\frac{13}{16}$, $\frac{29}{37}$ and $\frac{48}{59}$.

(7) What are the square roots of $5\frac{11}{49}$, $345\frac{24}{25}$, ϕ $15061\frac{119}{121}$ and $75628\frac{16993}{20736}$?

Answers: $2\frac{2}{7}$, $18\frac{3}{5}$, $122\frac{8}{11}$ and $275\frac{1}{144}$.

(8) Required the approximate square roots of 207, 97053, 187216 and 7429058.

Answers: 14. &c., 311. &c., 432. &c. and 2725. &c.

(9) Determine the square roots of 249.32, 876.535, 728.6527 and 29.41275 to four places of decimals.

Answers: 15.7898, 29.6063, 26.9935 and 5.4233.

(10) What are the square roots of the recurring decimals $.\dot{1}$ and $6.2\dot{4}\dot{9}$?

Answers: $.\dot{3}$ and $2.4\dot{9}$.

ϕ. Obs. $5\frac{11}{49} = 5+\frac{11}{49}$: & $345\frac{24}{25}$ is $345+\frac{24}{25}$ & so on.

EXTRACTION OF THE CUBE ROOT.

160. The Investigation of this operation is best conducted by general Symbols, and we shall merely put down here such observations and directions, as are necessary and sufficient for performing it.

Digits:

1, 2, 3, 4, 5, 6, 7, 8, 9:

Cubes:

1, 8, 27, 64, 125, 216, 343, 512, 729:

and it is important that these last numbers and the corresponding roots, should be committed to memory.

161. *Given the number of figures in any quantity, to find the number of figures in its cube-root.*

Since, the cube root of 1 is 1:
the cube root of 1000 is 10:
the cube root of 1000000 is 100: &c.,

it hence follows that the cube root of a number between 1 and 1000 consists of one figure: that of a number between 1000 and 1000000 of two figures: that of one between 1000000 and 1000000000 of three figures, and so on; so that if a point be placed over every third figure, beginning at the units' place, the number of points thus placed will manifestly be that of the digits in the cube root: and it is almost unnecessary to add, that the number of decimals in any quantity proposed must first be rendered a multiple of 3, and that it may then be pointed in the same manner, as is evident from what was said respecting the square root in article (159).

Rule for the Extraction of the Cube Root.

Point the figures as directed in the article above; and then, beginning at the left hand:

The root of the first period take,
And of the root a quotient make:
Which root must now a cube become,
To be the period taken from:
To the remainder then you must,
Bring down another period just:
Which being done, you then must see,
This number straight divided be,
By just three hundred times the square,
Of what the quotient figures are:

Let $a = \{$ Cube root of the 1st Period
$b =$ — 2nd Period
& so on $c =$ — 3rd
then $a^3 =$ Subtrahend of the 1st Period.
& the 2nd Divisor is — $3a^2.100 = 300a^2$.
& the 2nd Subtrahend is $3ab(a.10+b)10+b^3$
$= ab(a.10+b)30+b^3$
[illegible] $300 + ab^2.30 + b^3$
$+ 30ab^2 + b^3$
$+ 3ab^2.10 + b^3$.

The last squar'd, multipli'd by th' rest,
The product thirty times exprest:
The cube of the last figure too,
You must put in, if right, you do:
Add these, subtract them; so descend,
From point to point unto the end.

& the 3rd Divisor is — $3a^2.100\,00 = 30{,}000\,a^2$. or $3(a.10+b)^2.100 = 300(a.10+b)^2$,

& the 3rd Subtrahend is $3c(a.10+b)^2 100 + 3c^2(a.10+b) + c^3 = c(a.10+b)^2 300 + c^2(a.10+b)30 + c^3$.

Ex. Extract the cube root of 21952.

Here, after first pointing the numbers, we have

$$2\dot{1}95\dot{2}\,(28 = \text{cube root}:$$

$$a^3 = 2^3 = 8$$

Divisor

$$3a^2.100 = a^2 300 = 2^2 \times 300 = 1200\,)\,13952 \text{ dividend}:$$

Subtrahend $3a^2b.100 = 3\times4\times8\times100 = 9600$

$$3ab^2.10 = ab^2.30 = 8^2 \times 2 \times 30 = 3840$$

$$\& \ b^3 = 8^3 = 512$$

$$\& \ a^2b.300 + ab^2.30 + b^3 = 13952 \text{ subtrahend}:$$

and this is easily verified, for the cube of 28 = 21952.

The remark made before, respecting the trial of divisors, is applicable here; and the rule, which is very easily remembered, is adapted to vulgar fractions and decimals, exactly as that for the square root has been.

Before quitting this subdivision of the subject, we may take notice that the remainder at any step of the operation must not exceed three times the square of the corresponding quotient together with three times the quotient itself, and that the number of figures in the root may nearly be *doubled* by ordinary Division.

Examples for Practice.

(1) Determine the cube roots of 1331, 15625, 46656 and 117649.

Answers: 11, 25, 36 and 49.

(2) Find the cube roots of 2197, 185193, 704969 and 912673.

Answers: 13, 57, 89 and 97.

(3) What are the cube roots of 33076161, 15069223, 105823817 and 873722816?

Answers: 321, 247, 473 and 956.

(4) Determine the cube roots of 17.576, 132.651, 493.039 and 64481.201.

Answers: 2.6, 5.1, 7.9 and 40.1.

(5) Required the cube roots of 18.609625, 48.627125 and 122615.327232.

Answers: 2.65, 3.65 and 49.68.

(6) Extract the cube roots of $\frac{64}{343}$, $\frac{729}{140608}$, $49\frac{8}{27}$ and $7558\frac{197}{512}$.

Answers: $\frac{4}{7}$, $\frac{9}{52}$, $3\frac{2}{3}$ and $19\frac{5}{8}$.

(7) What are the approximate cube roots of 382.7, 21035.8, $.\dot{0}3\dot{7}$ and $1587.\dot{9}6\dot{2}$?

Answers: 7.26 &c., 27.604 &c., $.\dot{3}$ and $11.\dot{6}$.

EXTRACTION OF SOME OTHER ROOTS.

162. Though it is not intended here to enlarge upon the general methods of extracting the higher roots of numerical quantities, still the principles already developed may, by a little management, be rendered available to the discovery of several of them, as will be evinced in the following examples.

Ex. 1. Required the fourth root of 1679616.

The fourth power of any quantity being equivalent to the square of its square, it is evident that the fourth root of the quantity proposed will be the same as the square root of its square root, and may be found by the two following operations performed according to the rule laid down in article (159):

$$\begin{array}{r|l}
 & \dot{1}\,6\,\dot{7}\,9\,\dot{6}\,1\,\dot{6}\,(\,1296 \\
 & 1 \\
22) & 6\,7 \\
 & 4\,4 \\
249) & 2\,3\,9\,6 \\
 & 2\,2\,4\,1 \\
2586) & 1\,5\,5\,1\,6 \\
 & 1\,5\,5\,1\,6
\end{array}
\qquad\qquad
\begin{array}{r|l}
 & 1\,\dot{2}\,9\,\dot{6}\,(\,36 \\
 & 9 \\
66) & 3\,9\,6 \\
 & 3\,9\,6
\end{array}$$

and therefore the fourth root of 1679616, is 36.

Ex. 2. What is the sixth root of 308915776?

Here, the square root is found to be 17576: and the cube root of 17576 is 26, which is evidently the sixth root of the quantity proposed.

163. What has been done in these two instances will serve to shew that all higher roots of quantities may be extracted by the rules already given, whenever the reciprocals of the indices representing them can be resolved into the factors 2 and 3, or these factors repeated: thus, the *eighth* root of 21035.8 = the square root of the fourth root of 21035.8 = the square root of the square root of the square root of 21035.8 = 3.47032 &c.; but such a process manifestly cannot be made use of in other cases.

SURDS OR IRRATIONAL QUANTITIES.

164. Def. When the quantity whose root is to be extracted is not a *complete* square, cube, &c., we have seen that there will be a remainder left however far we may continue the operation, and the root can therefore be found only *approximately:* that is, such a quantity has no *exact* root, and its representation is termed a *Surd* or *Irrational Quantity.*

For instance, the square root of 2, expressed by $\sqrt{2}$, is evidently not a whole number, because the square of no whole number whatever is 2: neither can it be a vulgar fraction, because the square of every vulgar fraction properly so called is itself a vulgar fraction; and it cannot be a recurring decimal, because all such quantities are equivalent to finite vulgar fractions: in other words, the square root of 2 may be found as nearly as we please, but not exactly; and it is sometimes termed an *incommensurable* quantity, because it admits of no exact measure which is any finite quantity whatever, either integral or fractional.

165. The surds of most frequent occurrence are those designated by the sign $\sqrt[2]{\ }$ or $\sqrt{\ }$, or by the index $\frac{1}{2}$, and are termed *Quadratic* Surds: and in general, when any quantity is represented in the form of a surd by means of a *fractional index,* it is always understood that the numerator of the index denotes the power to which the number is intended to be raised, and that the deno-

minator expresses the root afterwards to be extracted: thus, $27^{\frac{2}{3}}$ is intended to represent the cube root of the square of 27, and is therefore equivalent to the cube root of 729, which is 9: that is, $27^{\frac{2}{3}}$, though expressed in the form of a surd, is in reality a rational quantity: and conversely.

166. Hence, the fundamental operations on surds must be performed upon their approximate values obtained as before: but these operations may frequently be shortened, as will appear in the following instances.

$$\text{Since } \sqrt{8} = \sqrt{4 \times 2} = \sqrt{4} \times \sqrt{2} = 2 \times \sqrt{2}, \text{ or } 2\sqrt{2};$$

$$\text{we have, in Addition, } \sqrt{8} + \sqrt{2} = 2\sqrt{2} + \sqrt{2} = 3\sqrt{2}:$$

$$\text{in Subtraction, } \sqrt{8} - \sqrt{2} = 2\sqrt{2} - \sqrt{2} = \sqrt{2}:$$

$$\text{in Multiplication, } \sqrt{8} \times \sqrt{2} = 2\sqrt{2} \times \sqrt{2} = 4:$$

$$\text{in Division, } \sqrt{8} \div \sqrt{2} = 2\sqrt{2} \div \sqrt{2} = 2:$$

where it is evident that the extraction of only one root is sufficient for the operations of Addition and Subtraction, and that both the product and quotient are rational quantities.

The Involution and Evolution of such quantities may frequently be effected in the same way: thus, the square of $2\sqrt{5}$ = the product of the squares of 2 and of $\sqrt{5} = 4 \times 5 = 20$, which is a rational number: and conversely.

Again, by multiplying each of the terms of the numerator and denominator by $\sqrt[3]{100}$, we have

$$\frac{\sqrt[3]{5.12} + \sqrt[3]{.03375}}{\sqrt[3]{80} - \sqrt[3]{.01}} = \frac{\sqrt[3]{512} + \sqrt[3]{3.375}}{\sqrt[3]{8000} - \sqrt[3]{1}}$$

$$= \frac{8 + 1.5}{20 - 1} = \frac{9.5}{19} = .5 = \frac{1}{2}, \text{ a rational quantity.}$$

167. It has been said that the values of surds may be found as nearly as we please: and this will clearly be done by continuing the extraction to any number of places of decimals in the root we may choose: thus, since $\sqrt{2} = 1.41421$ &c., we have

$$\begin{aligned}\sqrt{2} &= 1.4, \text{ nearly:}\\ &= 1.41, \text{ more nearly:}\\ &= 1.414, \text{ still more nearly:}\\ &= 1.4142, \text{ still more nearly:}\\ &= \&c. \ldots\ldots\ldots\ldots\ldots:\end{aligned}$$

and consequently its magnitude may be compared with that of any other numerical quantity either rational or irrational, although its absolute magnitude can never be exactly ascertained.

168. As quantities of this description have their origin in circumstances not purely *Arithmetical*, it is no objection to the definition of *Ratio* before given, that they scarcely seem to be included in it.

A ratio may however be incommensurable in *form*, though commensurable in *fact*, as is the case with $\sqrt{8} : \sqrt{2}$, whose magnitude is expressed by $2\sqrt{2} : \sqrt{2}$, or, by $2 : 1$.

Again, because $\sqrt{3} : \sqrt{2}$ is the same with the ratio $\sqrt{3} \times \sqrt{2} : \sqrt{2} \times \sqrt{2}$, or $\sqrt{6} : 2$; the magnitude of this ratio may be found to any degree of nicety, by increasing the number of decimal places in the extraction of the square root of 6.

The Arithmetic Mean between two numerical magnitudes being half their sum, will always be commensurable when they are so themselves; but the Geometric Mean, which is the square root of their product, will not necessarily be a terminating quantity under the same circumstances: thus, the Arithmetic Mean between 13 and 24 is 18.5, a rational quantity; whereas the Geometric Mean between them is $\sqrt{312} = 17.663$ &c., which is an incommensurable magnitude.

Examples for Practice.

(1) Find the approximate values of $4 \times \left(\frac{5}{156}\right)^{\frac{1}{3}}$, and $\sqrt{3} \times (\sqrt{5} - 1)$, to four places of decimals.

Answers: .7161, and 2.1409.

(2) What are the sum and difference of $5\sqrt{2}$ and $7\sqrt{8}$?

Answers: 26.8698 &c., and 12.7278 &c.

(3) Find the value of the compound expression $16\sqrt{3} + 10\sqrt[3]{4} - 4\sqrt{12} - 3\sqrt[3]{108}$.

Answer: 15.43 nearly.

(4) Determine the product and quotient of $5\sqrt{18}$ and $7\sqrt{63}$.

Answers: 1178.415 &c. and .381 &c.

(5) What is the square of $3\sqrt{7}$, and the cube of $\sqrt{2} \times \sqrt[3]{9}$?

Answers: 63, and $18\sqrt{2}$.

(6) Required the approximate values of the square roots of $\sqrt{11}$, and $14 - 6\sqrt{5}$.

Answers 1.82 &c., and .763 &c.

(7) Which is the greater of $\sqrt{2} + \sqrt{7}$ and $\sqrt{3} + \sqrt{5}$: also, of $\sqrt{6} - \sqrt{5}$ and $\sqrt{8} - \sqrt{7}$?

Answers: $\sqrt{2} + \sqrt{7}$, and $\sqrt{6} - \sqrt{5}$.

(8) Reduce $\sqrt{20}$, $2\sqrt{45}$ and $3\sqrt{80}$, so that they may contain the same surd.

Answer: $2\sqrt{5}$, $6\sqrt{5}$ and $12\sqrt{5}$.

(9) Determine the exact value of $\sqrt{19 + 8\sqrt{3}} + \sqrt{19 - 8\sqrt{3}}$.

Answer: 8.

(10) Find the exact value of the compound surd $\sqrt[3]{\sqrt{108} + 10} - \sqrt[3]{\sqrt{108} - 10}$.

Answer: 2.

The subjects of this chapter are fully discussed by means of general symbols in the Author's *Elements of Algebra*.

CHAPTER VIII.

THE NATURE AND PROPERTIES OF LOGARITHMS.

169. Def. 1. *Logarithms* are a series of magnitudes increasing by a common *Difference*, corresponding to another series of magnitudes increasing by a common *Multiplier:* thus, if the former series be the natural numbers increasing by the common difference 1, as

$$0, 1, 2, 3, 4, 5, 6, 7, 8, 9, \&c.;$$

and the latter begin with 1, and increase by the common multiplier or factor 2, as

$$1, 2, 4, 8, 16, 32, 64, 128, 256, 512, \&c.,$$
$$\text{or, } 2^0, 2^1, 2^2, 2^3, 2^4, 2^5, 2^6, 2^7, 2^8, 2^9, \&c.;$$

any term of the former series is *defined* to be the logarithm of the corresponding term in the latter: thus, we have

$$\begin{array}{ll} 0 = \log \text{ of } 2^0 \text{ or } 1; & 5 = \log \text{ of } 2^5 \text{ or } 32; \\ 1 = \log \text{ of } 2^1 \text{ or } 2; & 6 = \log \text{ of } 2^6 \text{ or } 64; \\ 2 = \log \text{ of } 2^2 \text{ or } 4; & 7 = \log \text{ of } 2^7 \text{ or } 128; \\ 3 = \log \text{ of } 2^3 \text{ or } 8; & 8 = \log \text{ of } 2^8 \text{ or } 256; \\ 4 = \log \text{ of } 2^4 \text{ or } 16; & 9 = \log \text{ of } 2^9 \text{ or } 512; \end{array}$$
$$\&c.;$$

where the number 2, which has been arbitrarily assumed, is called the *Radix* or *Base* of the *System* of Logarithms: and it is evident, that if the magnitude of any term in either of these series of quantities be assigned, that of the corresponding term in the other will be given.

Also, if an arithmetic mean between any two of the terms of the former series be found, it is manifest, from the manner in which the two series are connected, that a geometric mean between the two corresponding terms of the second series must have the same relation to it, throughout the whole extent of both the series adopted.

A simpler idea of these numbers will perhaps be had by defining the logarithm of a magnitude to be the index

of a certain *fixed* number, which, when raised to the power denoted by that index, produces the magnitude; the fixed number being assumed of any magnitude whatever, that of unity excepted, because every power of 1 is 1.

170. Def. 2. If the number 10, which is the Base of the Common System of Notation, be adopted for the base of the logarithms as above defined, it is evident that the terms of the series

$$0,\ 1,\ 2,\ 3,\ 4,\ 5,\ 6,\ 7,\ 8,\ 9,\ \&c.,$$

will, by the last article, be the logarithms of the corresponding terms of the series

$$10^0,\ 10^1,\ 10^2,\ 10^3,\ 10^4,\ 10^5,\ 10^6,\ 10^7,\ 10^8,\ 10^9,\ \&c.:$$

that is, in a system of logarithms whose base is 10,

$$\begin{aligned}
0 &= \log 10^0 \text{ or } 1\ ; \\
1 &= \log 10^1 \text{ or } 10\ ; \\
2 &= \log 10^2 \text{ or } 100\ ; \\
3 &= \log 10^3 \text{ or } 1000\ ; \\
4 &= \log 10^4 \text{ or } 10000\ ; \\
5 &= \log 10^5 \text{ or } 100000\ ; \\
6 &= \log 10^6 \text{ or } 1000000\ ; \\
7 &= \log 10^7 \text{ or } 10000000\ ; \\
8 &= \log 10^8 \text{ or } 100000000\ ; \\
9 &= \log 10^9 \text{ or } 1000000000\ ; \\
\&c. &= \&c.\ ;
\end{aligned}$$

and it is further manifest, from what has been said, that the arithmetic mean between any two terms of the first series, will be the logarithm of the geometric mean between the two corresponding terms of the second.

The arithmetic mean of 0 and 1 is .5: $= \frac{1}{2} = .5$

the geometric mean of 1 and 10 is 3.16227 &c.;

and therefore .5 = logarithm of 3.16227 &c.

The arithmetic mean of .5 and 1 is .75: $= \frac{3}{4} = .7$

the geometric mean of 3.16227 &c. and 10 is 5.62341 &c.;

whence .75 = logarithm of 5.62341 &c.

The arithmetic mean of 1 and 2 is 1.5: $= \frac{3}{2}$

the geometric mean of 10 and 100 is 31.62277 &c.;

whence 1.5 = logarithm of 31.62277 &c.:

and by continued repetitions of the process upon these and other numbers, it follows that the logarithms of all magnitudes whatever might be ascertained, though the labour requisite to do it would be immense. It appears moreover, that 0 is the logarithm of 1 in every system, whatever its base may be.

171. Def. 3. There is no difficulty in seeing that the logarithm of any magnitude between 1 and 10 will be a decimal fraction: that of any magnitude between 10 and 100 will be 1, with a decimal fraction annexed: that of one between 100 and 1000 will be 2, with a corresponding decimal fraction, and so on: for we have seen that

$$\frac{1}{2} = 0.5 = \log 3.16227 \text{ \&c.}:$$
$$\frac{3}{4} = 0.75 = \log 5.62341 \text{ \&c.}:$$
$$\frac{3}{2} = 1.5 = \log 31.62277 \text{ \&c.}:$$

and the integers 0, 1, 2, 3, &c., to the left of the decimal, points in the logarithms of all magnitudes, are called the *Characteristics* of those logarithms: thus, 0 is the characteristic of the logarithms of all magnitudes between 1 and 10; 1 is the characteristic of the logarithms of all magnitudes between 10 and 100; 2 that of all magnitudes between 100 and 1000; &c.

172. Def. 4. If the logarithms of all magnitudes be calculated by processes analogous to the one above explained, (or indeed by any other methods which the present advanced state of mathematical science may suggest, but which were unknown to the more early writers upon the subject,) and the results be put into the form of a table, we shall have what is called a *Table of Logarithms*; and this may be used to facilitate the arithmetical operations of Multiplication, Division, Involution and Evolution, and to render these operations, when applied to surds, or other complicated magnitudes, exceedingly concise and easy. The advantages thus conferred upon the practical mathematician will be fully explained and exemplified in the following articles.

173. *The Logarithm of the Product of two magnitudes is equal to the sum of the Logarithms of those magnitudes.*

Resuming the two series of magnitudes last used, we have

logarithms, 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, &c.:
numbers, $1, 10^1, 10^2, 10^3, 10^4, 10^5, 10^6, 10^7, 10^8, 10^9$, &c.:
and in these we observe that

$$\begin{aligned}
\log (1 \times 10) &= \log 10 = 1 = 0 + 1 \\
&= \log 1 + \log 10: \\
\log (10 \times 100) &= \log 1000 = 3 = 1 + 2 \\
&= \log 10 + \log 100: \\
\log (10 \times 1000) &= \log 10000 = 4 = 1 + 3 \\
&= \log 10 + \log 1000: \\
\log (100 \times 10000) &= \log 1000000 = 6 = 2 + 4 \\
&= \log 100 + \log 10000: \\
\&c. &= \&c.:
\end{aligned}$$

also, it is manifest, from the formation of these numbers, that the same must hold universally true, and that

$$\begin{aligned}
\log 6 &= \log (2 \times 3) = \log 2 + \log 3: \\
\log 15 &= \log (3 \times 5) = \log 3 + \log 5: \\
\log 24 &= \log (4 \times 6) = \log 4 + \log 6: \\
\&c. &= \&c.:
\end{aligned}$$

and this property may be rendered available to facilitate the multiplication of numbers whenever a table of logarithms, as explained in the last article, is at hand.

Ex. Let it be required to find the product of the numbers 7 and 23, by means of a table of logarithms.

Here, referring to tables of this description, we find

$$\log 7 = 0.8450980,$$
$$\log 23 = 1.3617278,$$

the characteristics which are there omitted, being 0 and 1 respectively, for the reasons assigned in article (171); whence, the logarithm of the required product will be

$$0.8450980 + 1.3617278 = 2.2068258;$$

and by looking again into the table, we find that this quantity without the characteristic, namely, .2068258 is the logarithm of 161, the characteristic itself merely shewing that the number is between 100 and 1000: that is, we have now the logarithm of the required product equal to the logarithm of 161; and, consequently, the product itself will be 161.

The operation above given may, by means of the arithmetical signs, be more conveniently expressed as follows:

$$\begin{aligned}\log(7 \times 23) &= \log 7 + \log 23\\ &= 0.8450980 + 1.3617278\\ &= 2.2068258\\ &= \log 161:\end{aligned}$$

and therefore $7 \times 23 = 161$, as we know to be the case.

Precisely in the same manner, whatever be the number of factors, as 17, 26, 35, &c., we shall have

$$\begin{aligned}\log(17 \times 26 \times 35 \times \&c.) &= \log 17 + \log(26 \times 35 \times \&c.)\\ &= \log 17 + \log 26 + \log 35 + \&c.,\end{aligned}$$

from which the product may be ascertained, as in the preceding example.

174. *The Logarithm of the Quotient of two magnitudes is equal to the difference of the Logarithms of those magnitudes.*

Referring to the statement made at the head of the last article, we see that

$$\begin{aligned}\log(10 \div 1) &= \log 10 = 1 = 1 - 0\\ &= \log 10 - \log 1:\\ \log(1000 \div 10) &= \log 100 = 2 = 3 - 1\\ &= \log 1000 - \log 10:\\ \log(1000000 \div 100) &= \log 10000 = 4 = 6 - 2\\ &= \log 1000000 - \log 100:\\ \&c. &= \&c.:\end{aligned}$$

and the general nature of these quantities leads us to conclude similarly, that

$$\begin{aligned}\log 3 &= \log(6 \div 2) = \log 6 - \log 2:\\ \log 9 &= \log(27 \div 3) = \log 27 - \log 3:\\ \log 23 &= \log(161 \div 7) = \log 161 - \log 7\\ \&c. &= \&c.\end{aligned}$$

This property will enable us to ascertain the quotient of two quantities, merely by the help of a logarithmic table.

Ex. What is the quotient arising from the division of 324 by 27?

Here, we shall have immediately,

$$\begin{aligned}\log(324 \div 27) &= \log 324 - \log 27\\ &= 2.5105452 - 1.4313639\\ &= 1.0791813\\ &= \log 12:\end{aligned}$$

whence it follows, from the equality of these logarithms, that

$$324 \div 27 = 12,$$

as is easily verified by ordinary division.

175. *The Logarithm of the Power of a magnitude is equal to the Logarithm of that magnitude multiplied by its index.*

For, we have seen in the preceding articles, that

$$\begin{aligned}\log 10^2 &= \log 100 = 2 = 2 \times 1 = 2 \times \log 10:\\ \log 10^3 &= \log 1000 = 3 = 3 \times 1 = 3 \times \log 10:\\ \log 10^4 &= \log 10000 = 4 = 4 \times 1 = 4 \times \log 10:\\ \&c. &= \&c.:\end{aligned}$$

and similar conclusions will manifestly hold of the powers of any other magnitudes, as

$$\begin{aligned}\log 4^3 &= 3 \times \log 4:\\ \log 9^7 &= 7 \times \log 9:\\ \log 18^{10} &= 10 \times \log 18:\\ \&c. &= \&c.\end{aligned}$$

Ex. To find the seventh power of 2, we have

$$\begin{aligned}\log 2^7 &= 7 \times \log 2\\ &= 7 \times 0.3010300\\ &= 2.1072100 = \log 128:\end{aligned}$$

whence, suppressing the logarithms of both, we have

$$2^7 = 128,$$

as is easily shewn to be true.

176. *The Logarithm of the Root of a magnitude is equal to the Logarithm of that magnitude divided by the whole number which denotes the root.*

For, as before, it is evident that

$$\begin{aligned}\log \sqrt{100} &= \log 10 = 1 = 2 \div 2 = \tfrac{1}{2} \log 100:\\ \log \sqrt[3]{1000} &= \log 10 = 1 = 3 \div 3 = \tfrac{1}{3} \log 1000:\\ \log \sqrt[5]{100000} &= \log 10 = 1 = 5 \div 5 = \tfrac{1}{5} \log 100000:\\ \&c. &= \&c.:\end{aligned}$$

and similarly, whatever the numbers may be, as

$$\log \sqrt{11} = \tfrac{1}{2} \log 11:$$
$$\log \sqrt[4]{125} = \tfrac{1}{4} \log 125:$$
$$\log \sqrt[9]{3421} = \tfrac{1}{9} \log 3421:$$
$$\&c. = \&c.$$

Ex. To extract the seventh root of 128, we have

$$\begin{aligned}\log \sqrt[7]{128} &= \tfrac{1}{7} \log 128\\ &= \tfrac{1}{7}(2.1072100)\\ &= .3010300 = \log 2:\end{aligned}$$

whence is immediately obtained $\sqrt[7]{128} = 2$,

as appears also from the example of the last article.

177. From the preceding articles and examples given to illustrate them, we perceive that by the assistance of a table of logarithms, the operation of *Multiplication* is reduced to that of *Addition:* the operation of *Division* to that of *Subtraction:* the operation of *Involution* to that of *Multiplication,* and the operation of *Evolution* to that of *Division:* and it cannot now be difficult to see of what immense importance such numbers must be in those departments of science wherein these operations are called into frequent practice, and more particularly in the use of surds or other very complicated quantities, which it would require great labour to treat according to the rules previously laid down.

178. As far as the *theoretical* view of logarithms is concerned, it is manifestly of very little importance what magnitude be adopted as the base of the system: but, in *practice,* the one here assumed may easily be shewn to possess great advantages over all others, both as to the computations of the numbers themselves, as well as to their practical use.

From the properties of these numbers taken notice of in articles (173) and (174), it will appear that the logarithms of all magnitudes expressed by the *same* significant digits, whether they be *integral, decimal* or *mixed,* differ only in their characteristics, the quantity to the right of the decimal point, sometimes called the *Mantissa,* remaining the same for them all.

Thus, from what was proved there, and from observing that by every multiplication or division of a quantity by the *Base*, the characteristic of its logarithm is increased or diminished by an *Unit*, we shall have

$$\begin{aligned}
\log 1230 &= \log (123 \times 10) \\
&= \log 123 + \log 10 = \log 123 + 1: \\
\log 12300 &= \log (123 \times 100) \\
&= \log 123 + \log 100 = \log 123 + 2: \\
\log 123000 &= \log (123 \times 1000) \\
&= \log 123 + \log 1000 = \log 123 + 3: \\
\&c. &= \&c.:
\end{aligned}$$

again,

$$\begin{aligned}
\log 12.3 &= \log (123 \div 10) \\
&= \log 123 - \log 10 = \log 123 - 1: \\
\log 1.23 &= \log (123 \div 100) \\
&= \log 123 - \log 100 = \log 123 - 2: \\
\log .123 &= \log (123 \div 1000) \\
&= \log 123 - \log 1000 = \log 123 - 3: \\
\&c. &= \&c.:
\end{aligned}$$

and since the characteristic of the logarithm of any figure in the place of units is 0, it is evident that the characteristic in any case will be *additive* or *subtractive*, according as the number is *greater* or *less* than unity: and it is on this account, that in the tables usually employed, the characteristics are entirely omitted, being intended to be supplied by the calculator when wanted.

Thus, by means of a logarithmic table we have

$$\log 123 = 2.0899052,$$

the characteristic 2 being here supplied from the considerations mentioned in (171): therefore, from what is done above, we get

$$\begin{aligned}
\log 1230 &= 1 + \log 123 = 3.0899052: \\
\log 12300 &= 2 + \log 123 = 4.0899052: \\
\log 123000 &= 3 + \log 123 = 5.0899052: \\
\log 1230000 &= 4 + \log 123 = 6.0899052: \\
\&c. &= \&c. \qquad = \&c.:
\end{aligned}$$

$$\begin{aligned}
\log 12.3 &= \log 123 - 1 = 1.0899052: \\
\log 1.23 &= \log 123 - 2 = 0.0899052: \\
\log .123 &= \log 123 - 3 = \bar{1}.0899052: \\
\log .0123 &= \log 123 - 4 = \bar{2}.0899052: \\
\&c. &= \&c. \qquad\quad = \&c.:
\end{aligned}$$

the small lines made over the 1 and 2 in the last two logarithms being intended to shew that the characteristic is there to be subtracted, instead of being added as in the rest, the mantissa still remaining additive as before.

The construction of logarithmic tables will consequently be much facilitated by the adoption of the number 10 as their base, a single mantissa now belonging to all magnitudes expressed by the same significant digits, which evidently could not be the case were any other assumed in its stead: and the advantage arises entirely from the circumstance of this number being the base of the system of notation in general use.

179. It would be foreign to the design of the present work, to enter into the detail of the methods employed in the construction of a Table of Logarithms, and we shall merely notice, among some of the uses of such a table, how the logarithms of numbers may, in certain cases, be derived from one another, and what expedients may be resorted to in order to establish their correctness.

180. *To find the Logarithm of a Composite Number.*

Let the number be decomposed into its prime factors; then by article (173), it is evident that the logarithm of the number proposed is equal to the sum of the logarithms of all its factors.

Thus, since $987 = 3 \times 329 = 3 \times 7 \times 47$,

we have $\log 987 = \log 3 + \log 7 + \log 47$;

and if the latter be known, the first is found: also, these logarithms, if calculated independently, will verify one another.

181. *To find the Logarithm of a Fraction.*

Let the logarithm of the denominator be subtracted from the logarithm of the numerator, and the difference will be the logarithm of the proposed fraction, as appears from article (174).

$$\text{Thus, } \log \frac{5}{7} = \log 5 - \log 7:$$

$$\text{and } \log 3\tfrac{4}{5} = \log \frac{19}{5} = \log 19 - \log 5:$$

and from these instances it follows that the logarithm of a *proper* fraction is subtractive, whilst that of an *improper* fraction is additive.

In practice, the logarithm of a proper fraction is adjusted so as to have its mantissa *additive* and its characteristic *subtractive,* as in article (178): thus,

$$\log \frac{5}{7} = \log 5 - \log 7 = \log 50 - 1 - \log 7 = \bar{1} + (\log 50 - \log 7).$$

182. *To find the Logarithm of a Power or a Surd.*

Multiply the logarithm of the quantity by its index, whether *integral* or *fractional,* and the result will be the logarithm of the power or surd proposed, as is evident from articles (175) and (176).

$$\text{Thus, } \log 7^2 = 2 \log 7:$$

$$\text{and } \log \left(\frac{2}{9}\right)^{\frac{3}{5}} = \frac{3}{5} \log \frac{2}{9} = \frac{3}{5} (\log 2 - \log 9):$$

and the logarithm of a surd will therefore be greater or less than the logarithm of its root, according as the index is greater or less than 1.

183. *To find a fourth proportional to three given magnitudes,*

From the sum of the logarithms of the second and third magnitudes, subtract that of the first, and the remainder will be the logarithm of the fourth proportional, which may therefore be found by the tables.

Thus, let x be a fourth proportional to 3, 7 and 11: then since

$$3 : 7 :: 11 : x,$$

we have

$$x = \frac{7 \times 11}{3};$$

and therefore $\log x = \log 7 + \log 11 - \log 3.$

184. *To find a mean proportional, or geometric mean between two given magnitudes.*

Divide the sum of the logarithms of the proposed quantities by 2, and the quotient will be the logarithm of their mean proportional.

Thus, if x be the mean proportional between 13 and 17, we have, by the definition of a mean proportional,

$$13 : x :: x : 17;$$

and therefore by article (122), we obtain

$$x^2 = 13 \times 17, \text{ and } x = (13 \times 17)^{\frac{1}{2}};$$

whence, $\log x = \frac{1}{2} (\log 13 + \log 17)$.

185. The preceding articles shew us that, in the formation of a set of Logarithmic Tables, it will be necessary to calculate the logarithms of the prime numbers only; and that those of their various multiples may then be found by addition.

When part of a table has thus been constructed, one portion of it may be used to verify another: thus, when we have found the logarithms of 3, 5 and 6, we should have

$$1 = \log 10 = \log \frac{30}{3} = \log 30 - \log 3 = \log 5 + \log 6 - \log 3:$$

and, by means such as these, a check may be applied at any stage of the process, in order to ascertain the correctness of the previous computations.

186. For the reader's exercise, we have put down here, the logarithms of all the prime numbers less than 100, without their characteristics; and he will thus be enabled to construct for himself a table of the logarithms of all other numbers up to 100.

Nos.	Logarithms.	Nos.	Logarithms.
2	3010300	43	6334685
3	4771213	47	6720979
7	8450980	53	7242759
11	0413927	59	7708520
13	1139434	61	7853298
17	2304489	67	8260748
19	2787536	71	8512583
23	3617278	73	8633229
29	4623980	79	8976271
31	4913617	83	9190781
37	5682017	89	9493900
41	6127839	97	9867717

These logarithms are extracted from Mr Babbage's Tables, which every *practical* Student should have in his possession.

Examples for Practice.

(1) Required the logarithms of 5 and 168.

Answers: .6989700, and 2.2253093.

(2) Determine the logarithms of 1.04 and 3690.

Answers: .0170334, and 3.5670265.

(3) What are the logarithms of $1\frac{3}{4}$ and $\frac{8}{11}$?

Answers: .2430380, and $\bar{1}.8616973$.

(4) Express the logarithm of 225 by means of the logarithms of 2 and 3, and verify it.

Answer: 2 – 2 log 2 + 2 log 3.

(5) Given the logarithms of 3 and 7, find the logarithm of 14700, and verify it.

Answer: 2 + log 3 + 2 log 7.

(6) Given the logarithms of 2 and 3, deduce the logarithm of .0072, and prove the converse.

Answer: 3 log 2 + 2 log 3 – 4.

(7) Find the logarithm of 50000 in terms of the logarithms of 216 and .081.

Answer: $5\frac{3}{4} - \frac{1}{3}\log 216 + \frac{1}{4}\log .081.$

(8) Express the logarithms of 8 and 9 in terms of those of 6 and 15.

Answers: $\frac{3}{2} + \frac{3}{2}\log 6 - \frac{3}{2}\log 15$, and $\log 6 + \log 15 - 1$.

(9) Find the logarithm of 83349, from the logarithms of 3 and .21.

Answer: $6 + 2\log 3 + 3\log .21.$

(10) Given the logarithms of 15 and 16, find those of 27 and $4\frac{1}{20}$.

Answers: $3\log 15 + \frac{3}{4}\log 16 - 3$, and $4\log 15 + \frac{3}{4}\log 16 - 5$.

(11) Find the logarithms of 15.625 and .00475.

Answers: $6\log 5 - 3$, and $2\log 5 + \log 19 - 5$.

(12) Required the logarithms of $\frac{9}{16}$ and $\frac{2}{375}$ in terms of the logarithms of 2, 3 and 5.

Answers: $2\log 5 + 2\log 3 - 2\log 2 - 2$,
and $4\log 2 - \log 3 - 3$.

(13) Determine the logarithms of $\sqrt[3]{\frac{24}{135}}$ and $\sqrt[4]{1.625}$, by means of those of 2, 3, 5 and 13.

Answers: $2\log 2 - \frac{2}{3}\log 3 + \frac{2}{3}\log 5 - 1$,

and $\frac{1}{4}\log 13 - \frac{3}{4}\log 2$.

(14) Express the logarithm of 7 in terms of the logarithms of 2 and $.\dot{7}1428\dot{5}$.

Answer: $1 - \log 2 - \log .\dot{7}1428\dot{5}$.

(15) Given the logarithm of $10424 = 4.0180353$: find the fifth root of $1\frac{3}{13}$.

Answer: 1.0424.

(16) Determine the value of the expression $\frac{2^6 \times 2.5^3}{4^3 \times 10^6}$ by means of logarithms.

Answer: 6.25.

(17) Find a fourth proportional to the quantities 1.3, .0104 and 2.375 by logarithms.

Answer: .019.

(18) Determine by logarithms a mean proportional between the magnitudes .004 and 72250.

Answer: 17.

(19) Given $.200686 = \log 1.58740 = 2 \log 1.25992$: find the value of $\sqrt[3]{4} - \sqrt[3]{2}$.

Answer: .32748.

(20) Given $2.2309306 = \log 170.188$: it is required to find the value of $8 \times \sqrt[5]{7\sqrt{2} \times \sqrt[3]{3}}$.

Answer: 13.61504.

(21) Required the number of figures in the product of 324 and 126, by means of logarithms.

Answer: 5.

(22) Find the numbers of digits in the results of the involutions of 2^{10} and 3^{12}, by means of logarithms.

Answers: 4, and 6.

(23) Required by a table of logarithms, the index of 5 which shall give a result equal to 20.

Answer: $\frac{1 + \log 2}{1 - \log 2}$.

(24) Find the logarithm of 180 in a system whose base is 12, by means of a table of common logarithms.

Answer: $\frac{1 + \log 2 + 2 \log 3}{2 \log 2 + \log 3}$.

(25) Shew that the *Mantissa* of a logarithm depends upon the figures, and not upon the pointing off: and that the *Characteristic* depends upon the pointing off, and not upon the figures.

The invention of Logarithms is due to the celebrated JOHN NAPIER or NEPER, Baron of *Merchiston* in

Scotland, who was born in the year 1550, and died in the 68th year of his age. The base of the *Napierian* System of Logarithms is the mixed magnitude 2.71828 &c.; but, for the great improvement in the subject hinted at in Article (178), we are indebted to Mr HENRY BRIGGS, Professor of Geometry at *Oxford,* by whom a table was published in the year 1624.

The reader, who may be desirous of further information upon this interesting portion of science, is referred to Dr HUTTON'S Mathematical Tables, which contain an account of the discoveries of the most celebrated writers, connected with it: but he will not be able to appreciate their ingenuity and merits without a much more extensive knowledge of numerical calculations than can be acquired from this or any other treatise on *Arithmetic.*

CHAPTER IX.

THE APPLICATION OF ARITHMETIC TO GEOMETRY.

187. Def. 1. In several of the preceding chapters, the symbols and signs of *Pure Arithmetic* have been transferred from *abstract* numerical magnitudes, so as to represent the relations between *concrete* or *particular* quantities arithmetically considered; and it is on the same principle that the objects of *Geometry* or *Geometrical Magnitudes*, as *Lines* or *Distances*, *Superficies* or *Areas*, and *Solid Contents* or *Volumes*, are valued or compared by means of the numbers representing their respective *Dimensions*: also, a line having *length* only, is considered as possessed of only *one* dimension: a superficies, having both *length* and *breadth*, comprises *two* dimensions; and a solid has *three* dimensions, inasmuch as it is defined by means of three magnitudes, *length*, *breadth*, and *depth* or *thickness*.

188. Def. 2. A *Measure* in Geometry, is a certain magnitude assumed as an *Unit*, with which other magnitudes of the *same kind* may be compared: and though one magnitude neither contains another, nor is contained in it an *exact* number of times, there may still be a third and smaller magnitude which is capable of *measuring* them both. A *measure* thus defined has therefore the same relation to *quantity*, as *unit* or 1 has to *number;* and all quantities and numbers are said to be *equal to* the aggregates or sums of their measures and units respectively.

It appears, therefore, that when the magnitudes of lines are once numerically expressed, the *Principles of Geometry* must themselves furnish the means of valuing, or comparing with each other, those of both superficies and solids, of which lines naturally form the dimensions: and on this account we shall first establish the *Theory* of *Lineal Measure*, and then deduce those of *Superficial* and *Solid Measure* from it.

THE THEORY OF LINEAL OR LONG MEASURE.

189. **Def.** An *Unit* of lineal or long measure, is a straight line of a certain length, *arbitrarily* fixed upon; and by the determination of the *ratios* which other lines bear to it, we are enabled to compare with each other, all magnitudes of this description.

Thus, if the straight line ab be considered the lineal unit, the numerical magnitude of the straight line AB will manifestly be

determined from the following proportion:

the magnitude of ab : the magnitude of AB :: *a* lineal unit : *the* lineal units in AB; that is,

the numerical magnitude of AB will be

$$= \text{the magnitude of } ab \times \frac{\text{the lineal units in } AB}{\text{a lineal unit}}$$

= the magnitude of ab × the *number* of lineal units in AB: whence, representing the magnitude of ab by *unity* or 1, we shall have the numerical magnitude of AB represented by the *number* of lineal units contained in it; that is, if the lengths of two straight lines AB and CD be respectively 4 times and 6 times as great as the length of the lineal unit ab, the corresponding magnitudes of the lines AB and CD will be 4 and 6 respectively, which are expressed by the equalities

$$AB = 4, \text{ and } CD = 6:$$

and a similar method of proceeding will shew that, if any straight line whatever be a *multiple* of the lineal unit, the numerical representative of its magnitude must be a *whole* number.

Next, if the proposed line AB be not an *exact* multiple of the lineal unit ab, but have a *common* measure with it, so that, when they are *both* divided by it, the common measure is contained 7 times and 3 times in them respectively; then, we shall evidently have

the magnitude of AB : the common measure :: 7 : 1;

and the common measure : the magnitude of ab :: 1 : 3;

that is, by Articles (122) and (123), we have

the magnitude of AB = 7 × the common measure;

and the common measure $= \frac{1}{3} \times$ the magnitude of ab;

from which we conclude immediately, that

$$\text{the magnitude of } AB = 7 \times \frac{1}{3} \times \text{the magnitude of } ab$$
$$= \frac{7}{3} \times \text{the magnitude of } ab$$
$$= \frac{7}{3},$$

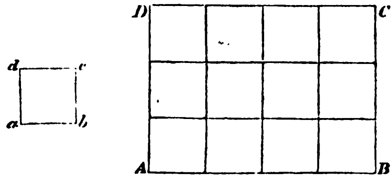
the magnitude of ab being represented by 1, as before; and thus we infer generally, that whenever a line is not a *multiple* of the lineal unit, but has a *common* measure with it, its magnitude may be represented by means of a *fraction*.

If, however, the proposed line AB be neither a multiple of the lineal unit ab, nor have any common measure with it, as, for instance, if $AB = \sqrt{2}$, then it is manifest that only an approximate arithmetical representation of it can be had, where the approximation may easily be carried far enough to answer every practical purpose, as appears from Article (167).

It need scarcely be observed here, that if the lineal unit be an *inch*, a *foot*, a *yard*, &c., the corresponding magnitudes of the proposed lines will be expressed in inches, feet, yards, &c., and their parts, respectively.

THE THEORY OF SUPERFICIAL OR SQUARE MEASURE.

190. DEF. An *Unit* of superficial or square measure is a square surface or area, whereof the length of each side is equal to that of the lineal unit: thus, if ab represent the *lineal* unit,

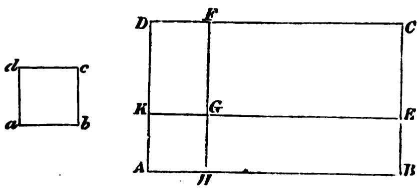


the square $abcd$ described upon it will be the *superficial* or *square* unit, having the *two* dimensions ab and ad,

which may be regarded as its *length* and *breadth:* and the magnitude of any proposed surface or area, will manifestly be obtained by finding what *multiple, part,* or *parts,* the surface or area is of *this* unit.

191. *The numerical representative of the Area of a rectangular parallelogram is equal to the product of those of two of its adjacent sides.*

Let $ABCD$ be a rectangular parallelogram, whereof the adjacent sides AB and AD contain 7 and 5 lineal units respectively; take $AH = AK =$ the *lineal* unit, and draw KE and HF parallel to AB and AD, intersecting in G, so that the square AG, being equal to the square



$abcd$, may represent the *superficial* unit: then by EUCLID, VI. 1, we have

the area of the parallelogram $ABEK$: the area of the superficial unit $AHGK$ :: $AB : AH$ :: $7 : 1$;

whence, by articles (122) and (123),

the area of the parallelogram $ABEK = 7 \times$ the area of the superficial unit $AHGK$;

again, by the same proposition, we have

the area of the parallelogram $ABCD$: the area of the parallelogram $ABEK$: $AD : AK$ :: $5 : 1$;

or, the area of the parallelogram $ABCD = 5 \times$ the area of the parallelogram $ABEK$;

and therefore, from the preceding equality, we obtain

the area of the parallelogram $ABCD = 5 \times 7 \times$ the area of the superficial unit $AHGK$;

whence, if the area of the superficial unit be represented by 1, the area of the parallelogram $ABCD$ will, on the same scale, be represented by

$$5 \times 7 \text{ or } 35,$$

which is the product of two of its adjacent sides;

or, the area of the parallelogram $ABCD = AB \times AD$

$$= 7 \times 5 = 35 \text{ superficial units.}$$

Also, from the general principle of the demonstration just given, it is evident that the same conclusion must hold good, if the sides be represented by *fractions* or *irrational quantities*, inasmuch as the proposition of geometry here made use of, has reference to *quantity*, and not to *number* only.

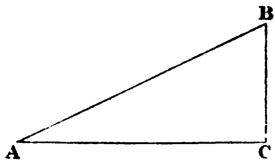
Ex. 1. Let the two sides of the rectangular parallelogram be equal to one another, and to 12 inches or 1 foot, so that $ABCD$ becomes a square: then the area of the *square* $ABCD$

$$= AB \times AD = 12 \times 12 = 144;$$

that is, if the side of a square contain 12 *lineal* inches, its area will comprise 144 *superficial* or *square* inches: or, in other words, 144 square inches are *equal* to 1 square foot.

Similarly, 9 square feet are equal to 1 square yard, and $30\frac{1}{4}$ or 30.25 square yards, to 1 square pole.

Ex. 2. Let the base AC, and the perpendicular altitude BC, of the triangle ABC right angled at C, be represented



according to the principles above explained, by 4 inches and by 3 inches respectively: then it follows, from Euclid, I. 47, that

$$AB^2 = AC^2 + BC^2$$

$$= 4^2 + 3^2 = 16 + 9 = 25:$$

whence, by extracting the square roots of both sides of the equality, we have

$$AB = 5 \text{ inches};$$

also, if the sides AC and BC were expressed in feet,

yards, &c., the corresponding value of AB would be found in those terms likewise.

If $AC = 3$ feet and $BC = 2$ feet, we shall have, by the same proposition,

$$AB^2 = AC^2 + BC^2 = 3^2 + 2^2 = 9 + 4 = 13:$$

and thence, by the extraction of the square root,

$$AB = \sqrt{13} = 3.605 \text{ \&c. feet},$$

which is only an approximation to the *true* value, but may be continued to as much nicety as we please.

If we had $AC = BC = 1$ yard, then would

$$AB^2 = AC^2 + BC^2 = 1^2 + 1^2 = 2:$$

and therefore, by performing the same operation, we have

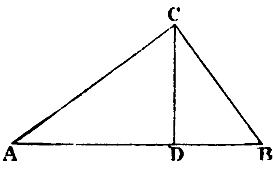
$$AB = \sqrt{2} = 1.4142135 \text{ \&c. yards}:$$

and this, in fact, proves the *hypothenuse* of a right-angled isosceles triangle, or the *diagonal* of a square, to be *incommensurable* with either of the *sides*.

From the last two instances, it appears that a quadratic surd may be expressed accurately in Geometry, though not so in Arithmetic; and it is also clear, from the mode of proceeding adopted, that any other *geometrical* proposition may be translated into the symbols of *Arithmetic*, and any part determined, when the number of the data is *sufficient* for the purpose.

192. From these principles, if the base and perpendicular altitude of a plane triangle be represented by numerical magnitudes, its area will be numerically represented by half their product.

For, let the base AB be equal to 8 feet, and the perpendicular altitude CD to 3 feet:



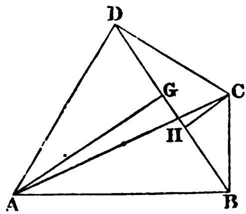
then, by EUCLID, I. 41, the area of the triangle ABC

is equal to *half* of the rectangular parallelogram whose base is AB and whose perpendicular altitude is CD: whence,

$$\text{the area of the triangle } ABC = \tfrac{1}{2} \text{ of } AB \times CD$$
$$= \tfrac{1}{2} \text{ of } (8 \times 3) = \tfrac{1}{2} \text{ of } 24 = 12;$$

that is to say, if the base and perpendicular altitude of a triangle be equivalent to 8 and 3 *lineal* units respectively, then will its area be represented by 12 *superficial* units of the same denomination; and it is of no consequence whether the dimensions be integral, fractional or irrational, as appears from Article (191).

193. If we take the four-sided figure $ABCD$, called a *trapezium*, and



find the lineal magnitudes of the *diagonal* BD, and of the *perpendiculars* AG and CH let fall upon it from the angles A and C, the area of the figure, being the sum of the areas of the two triangles ABD and BCD, may be ascertained.

Thus, if it be found that $BD = 5$, $AG = 4$, and $CH = 1\tfrac{1}{2}$ lineal units, respectively; we shall have

$$\text{the area of } ABCD = \text{the area of } ABD + \text{the area of } BCD$$
$$= \tfrac{1}{2} BD \times AG + \tfrac{1}{2} BD \times CH$$
$$= \tfrac{1}{2}(5 \times 4) + \tfrac{1}{2}(5 \times 1.5)$$
$$= \tfrac{20}{2} + \tfrac{7.5}{2} = 10 + 3.75$$
$$= 13.75 = 13\tfrac{3}{4} \text{ superficial units:}$$

and the same result must evidently have been obtained, if perpendiculars had been let fall upon the *other* diago-

nal AC, from the angles B and D, because the area of the *same* figure cannot have *two* different magnitudes.

Similarly, the area of any rectilineal figure may be found by adding together the areas of the triangles which compose it.

194. Conversely, if the area of a rectangular parallelogram, or of a triangle, and either its base or perpendicular altitude, be given, the other of these magnitudes will manifestly be obtained by *division*.

Also, if the superficial units comprised in the area of a square, whose base is AB, be 1521; it is evident that

$$AB^2 = 1521:$$

from which, by the extraction of the square root, we have

$$AB = 39:$$

that is, if the area of a square surface be 1521 *superficial* units, every one of its sides will be 39 *lineal* units, which may be inches, feet, yards, &c.

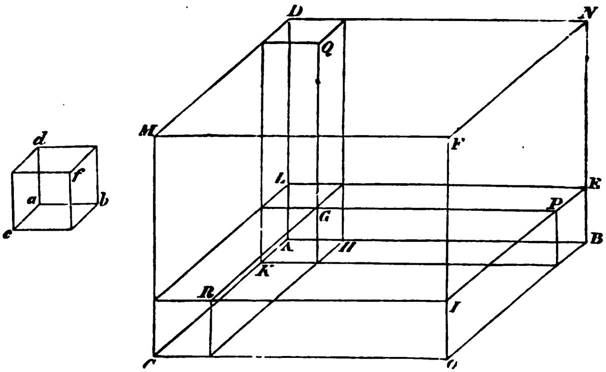
Again, an acre, being a rectangular parallelogram 40 poles in length and 4 poles in breadth, contains 4840 square yards, and will therefore be equal to a square whose side $= \sqrt{4840} = 69.57$ &c. $= 69\frac{4}{7}$ yards, nearly.

THE THEORY OF SOLID OR CUBIC MEASURE.

195. Def. An *Unit* of solid or cubic measure, is a cube or rectangular parallelopiped whose length, breadth and thickness are each equal in magnitude to the lineal unit; as the solid af represented below, wherein $ab = ac = ad =$ the lineal unit, which may be an inch, a foot, a yard, &c., as before, denotes the *solid* or *cubic* unit: and the solid content or volume of any other body of *three* dimensions will evidently be ascertained by finding what multiple, part or parts, it is of this unit, the lineal dimensions, or the length, breadth and thickness being supposed first to be numerically exhibited.

196. *The numerical representative of the Solid Content or Volume of a rectangular parallelopiped, is equal to the continued product of the magnitudes representing its length, breadth and thickness.*

Let $ABFM$ represent a rectangular parallelopiped, whereof the length $AB = 5$, the breadth $AC = 4$, and the thickness $AD = 3$ lineal units, the denominations of the dimensions being the same in each, whether inches, feet, yards, &c.:



take $AH = AK = AL =$ the lineal unit, and complete the construction as in the diagram; then it is manifest that AG will be a cube, whose magnitude is equal to that of the *solid* unit af; and, by EUCLID, XI. 25, we have

$$\text{the parallelopiped } AF : \text{the parallelopiped } AI$$
$$:: \square CD : \square CL :: AD : AL :: 3 : 1;$$

and the parallelopiped $AF = 3 \times$ the parallelopiped AI; also,

$$\text{the parallelopiped } AI : \text{the parallelopiped } AP$$
$$:: \square BI : \square BP :: AC : AK :: 4 : 1;$$

and the parallelopiped $AI = 4 \times$ the parallelopiped AP; again,

$$\text{the parallelopiped } AP : \text{the parallelopiped } AG$$
$$:: \square BL : \square HL :: AB : AH :: 5 : 1;$$

and the parallelopiped $AP = 5 \times$ the parallelopiped AG; whence, we have now, the parallelopiped AF

$= 3 \times$ the parallelopiped AI

$= 3 \times 4 \times$ the parallelopiped AP

$= 3 \times 4 \times 5 \times$ the parallelopiped AG;

but the parallelopiped AG being equal to the solid unit, is represented by 1; consequently, the numerical magni-

tude of the rectangular parallelopiped, whose three contiguous edges are equivalent to 3, 4 and 5 *lineal* units, will be represented by

$$3 \times 4 \times 5 = 60 \textit{ solid} \text{ units:}$$

that is, the content of the parallelopiped $ABFM$

$$= AB \times AC \times AD = 5 \times 4 \times 3 = 60.$$

If the three edges AB, AC, AD be all equal to one another, and their magnitude be 3 *lineal* feet or 1 yard, the parallelopiped becomes a cube, whose magnitude $= 3 \times 3 \times 3 = 27$ *solid* feet: that is, 27 solid or cubic feet are *equal* to 1 solid or cubic yard.

Similarly, 1728 cubic inches are equal to 1 cubic foot: and so on for other denominations.

197. Hence, also may be found the length of an edge of the cube which is of equal solid content with any proposed parallelopiped or solid, whose dimensions or volume are given.

Thus, if a parallelopiped be 7 inches in length, $3\frac{1}{2}$ inches in breadth, and $1\frac{3}{4}$ inches in depth, its solid content will be

$$7 \times 3\tfrac{1}{2} \times 1\tfrac{3}{4} = 42\tfrac{7}{8} \text{ cubic inches,}$$

which is therefore equal to the solid content of a cube whose edge

$$= \sqrt[3]{42\tfrac{7}{8}} = \sqrt[3]{42.875} = 3.5 = 3\tfrac{1}{2} \text{ lineal inches.}$$

In the same manner, the edge of a rectangular parallelopiped may be found by dividing the solid content by the area of the surface to which it is at right angles; and *vice versâ*.

198. It will not be necessary to pursue these subjects further in this place; and we shall here only insert directions for ascertaining the measures of such magnitudes as most frequently present themselves to our notice, without attempting their investigations, which more properly belong to other parts of Mathematics.

THE PRACTICE OF LINEAL MEASURE.

(1) *Right-angled Triangle.* The square root of the sum of the squares of the *sides* forming the right angle is equal to the *Hypothenuse:* and the square root of the difference of the squares of the hypothenuse and *either* side is equal to the *other* side.

(2) *Circle.* The circumference is equal to the product of twice the radius by 3.14159, nearly: and the radius is equal to the quotient of the circumference by 6.28318, nearly.

(3) Hence, the homologous lines in similar triangles, and in all circles, are proportional.

Ex. 1. If the base of a triangle be 1, and the perpendicular be 1, the hypothenuse $= \sqrt{1^2 + 1^2} = \sqrt{2}$.

If the base be $\sqrt{2}$, and the perpendicular be 1, the hypothenuse $= \sqrt{2 + 1} = \sqrt{3}$.

If the base be $\sqrt{3}$, and the perpendicular be 1, the hypothenuse $= \sqrt{3 + 1} = \sqrt{4} = 2$.

If the base be 2, and the perpendicular be 1, the hypothenuse $= \sqrt{4 + 1} = \sqrt{5}$, and so on: and in all these, only approximate *arithmetical* values of the surds can be found by evolution; also, it is worth noticing how all the primitive surds successively originate from these *geometrical* considerations, as has been hinted before at the end of Article (189).

Ex. 2. The wheels of a carriage are $2\frac{1}{2}$ yards asunder, and the inner wheel describes the circumference of a circle whose radius is 20 yards: find the difference of the paths of the two wheels.

The circumference of the inner circle $= 3.14159 \times 40$: the circumference of the outer circle $= 3.14159 \times 45$: whence, their difference will evidently $= 3.14159 \times 5 = 15.70795$ yards $= 15\frac{7}{10}$ yards, nearly.

Examples for Practice.

(1) Required the hypothenuse of a right-angled triangle whose sides are 24 and 32 feet.

Answer: 40 feet.

(2) Find the base of the right-angled triangle whose other sides are 4 and $\sqrt{48}$.

Answer: $4\sqrt{2}$.

(3) If a ladder 103.44 feet long, be placed so as to reach a window 40 feet high on one side of a street,

and a window 60 feet high on the other side: what is the breadth of the street?

Answer: 180 feet, nearly.

(4) Of two ships from the same port, one has sailed 50 leagues due east, and the other 84 leagues due north: what is their distance from each other?

Answer: $97\frac{3}{4}$ leagues, nearly.

(5) Find the circumference of a circle whose radius is 6.3662 yards.

Answer: 40 yards, nearly.

(6) If the diameter of the earth be 7912 miles, find the length of a French metre, which is one ten-millionth part of a fourth part of its circumference.

Answer: 39.37206 inches, nearly.

(7) Shew that $\frac{22}{7}$, $\frac{333}{106}$ and $\frac{355}{113}$ are approximations to the known numerical value of the circumference of a circle whose diameter is 1, and point out which is the nearest.

(8) Prove that $\frac{501 + 80\sqrt{10}}{240}$ is a close approximation to the semicircumference of a circle whose radius is represented by 1.

THE PRACTICE OF SUPERFICIAL MEASURE.

(1) *Parallelogram.* The area is equal to the product of the base and the perpendicular altitude.

(2) *Triangle.* The area is equal to half the product of the base and the perpendicular altitude.

(3) *Triangle.* From half the sum of the three sides, subtract each side separately: multiply together the half-sum and the three remainders, and the square root of the product will be equal to the area.

(4) *Trapezium.* The area is equal to half the product of either diagonal, and the sum of the perpendiculars let fall upon it, from the opposite angles.

(5) *Circle.* The area is equal to the square of the radius, multipled by 3.14159, nearly.

(6) *Sector.* The area is equal to half the product of the radius and the subtending arc.

(7) *Ellipse.* The area is equal to the product of the semi-axes, multiplied by 3.14159, nearly.

(8) Hence, the areas of similar plane figures are as the squares of their homologous lineal dimensions.

Ex. 1. Find the area of a triangle whose sides are 18, 24 and 30 poles.

Here, we have according to the directions above, half the sum of the three sides $= \frac{1}{2}(18 + 24 + 30) = 36$:

$$\text{also, } \left.\begin{aligned} 36 - 18 &= 18 \\ 36 - 24 &= 12 \\ 36 - 30 &= \ \ 6 \end{aligned}\right\} \text{ are the three remainders:}$$

whence, the area $= \sqrt{36 \times 18 \times 12 \times 6} = \sqrt{46656} = 216$ square poles.

Ex. 2. If the radius of a circle be 2 feet, find the side of the square whose area shall be equal to it.

The area of the circle $= 4 \times 3.14159 = 12.56636$ square feet, nearly: whence, by Article (194), the side of the required square $= \sqrt{12.56636} = 3.545$ feet, nearly.

Examples for Practice.

(1) If the sides of a triangle be 16.6, 18.32 and 28.6: find its area.

Answer: 143, nearly.

(2) If the diagonal of a trapezium be 498 yards, and the perpendiculars let fall upon it from the opposite angles be 10.8 and 18.8 yards: what is its area?

Answer: 7370.4 yards.

(3) Each side of a hexagon is 24 feet, and the perpendicular upon each side from a certain point within it is $12\sqrt{3}$ feet: find its area.

Answer: $864\sqrt{3}$ feet.

(4) Find the sides of the squares whose areas are 4970.25 square inches, and $885\frac{1}{16}$ square feet.

Answers: 70.5 inches, and $29\frac{3}{4}$ feet.

(5) How much must be cut off from a rectangular surface $2\frac{1}{4}$ feet broad, to make a square yard?

Answer: 4 feet.

(6) If two acres of land be laid out in the form of a circle, what is its radius?

Answer: $55\frac{1}{2}$ yards, nearly.

(7) Find the radius of a circle, whose area is equal to that of a square whose side is 5.317 yards.

Answer: 3 yards, nearly.

(8) The semiaxes of an ellipse are 25 and 49: find the radius of a circle of equal area.

Answer: 35.

(9) The base of a triangle is 14.1 yards, and its area is 64.86 yards: find its perpendicular height.

Answer: 9.2 yards.

(10) The side of an equilateral triangle is 6: find its area.

Answer: 15.588, nearly.

(11) The two equal sides of an isosceles triangle are 12 feet, and the base is 8 feet; required its area.

Answer: 45.2548 feet, nearly.

(12) Compare the area of a circle with the area of the square inscribed in it.

Answer: 3.14159 : 2, nearly.

(13) What is the relation between the area of a square, and that of the circle inscribed in it?

Answer: 4 : 3.14159, nearly.

(14) Required the area of the sector of a circle, whose arc and radius are each 2.57 inches.

Answer: 3.30245 inches.

(15) The radii of two concentric circles are 10 and 12 yards: find the space included between them.

Answer: 138.22996 yards, nearly.

(16) The areas of squares, circles, similar parallelograms and triangles, are as the squares of their homologous lineal dimensions.

THE PRACTICE OF SOLID MEASURE.

(1) *Parallelopiped.* The content is equal to the area of the base multiplied by the perpendicular height.

(2) *Prism and Cylinder.* The content is equal to the area of the base multiplied by the perpendicular height.

(3) *Pyramid and Cone.* The content is equal to the area of the base multiplied by one third of the perpendicular height.

(4) *Sphere or Globe.* The content is equal to the cube of the radius multiplied by 4.18879, nearly.

(5) Hence, the contents of similar solid bodies are as the cubes of their homologous lineal dimensions.

Ex. 1. Required the depth of a parallelopiped $29\frac{2}{3}$ long, and $44\frac{1}{2}$ broad, so that its content shall be equal to that of a cube whose edge is 89.

Here, the area of the base of the parallelopiped

$$= 29\tfrac{2}{3} \times 44\tfrac{1}{2} = \frac{89 \times 89}{3 \times 2} = \frac{89^2}{6}:$$

whence, the depth of the same solid

$$= 89^3 \div \frac{89^2}{6} = \frac{89^3 \times 6}{89^2} = 89 \times 6 = 534.$$

Ex. 2. The content of a cylinder is equal to the sum of the contents of a cone and hemisphere, having the same base and altitude.

Taking 1 to represent the radius of the hemisphere, we shall have immediately from the directions contained in this page:

the content of the hemisphere = 2.09439, nearly:

the content of the cone = 1.04719, nearly:

the content of the cylinder = 3.14159, nearly:

whence, we find the sum of the *two former*

= 3.14158 nearly, which is the *last,* very nearly:

and this would have been an *exact equality* were it not for the circumstance of *each* of the contents being only an *approximation* to its true value.

Examples for Practice.

(1) Each side of a square prism is 34 inches, and its height is $12\frac{5}{12}$ feet: how many solid feet does it contain?

Answer: 99 ft. 1172 in.

(2) A rectangular cistern whose length is $9\frac{3}{4}$ feet, and breadth 6 feet, contains $294\frac{1}{4}$ cubic feet: find its depth.

Answer: $5\frac{7}{234}$ feet.

(3) What length of a cylindrical stone roller 18 inches in diameter, must be taken to make 14.137155 solid feet?

Answer: 2 feet.

(4) The sides of the base of a triangular pyramid are 3, 4 and 5 feet, and its altitude is 6 feet: find its solid content.

Answer: 12 feet.

(5) The solid content of a sphere is two thirds of that of its circumscribed cylinder.

(6) A right cone, hemisphere and cylinder of the same base and altitude, are as the numbers 1, 2, 3.

(7) A sphere is equal to a cone, whose height is equal to the radius, and whose base is equal to four great circles of the sphere.

(8) The contents of cubes, spheres, similar parallelopipeds, cylinders and cones, are as the cubes of their homologous lineal dimensions.

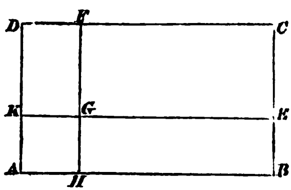
THE COMPUTATIONS OF ARTIFICERS.

199. DEF. *Artificers* generally take the dimensions of their work in *yards, feet, inches, parts, &c.:* and it is usual to reduce the yards to feet, so that the different denominations are *all* connected by the same number 12, or decrease in a *twelvefold* ratio, from the place of feet towards the right hand. For the sake of uniformity, the denominations after feet are termed *primes, seconds, thirds, &c.*, distinguished respectively by accents $'$, $''$, $'''$, &c., placed a little to the right, contiguous to the figures to which they belong: thus, 20 feet, 8 inches, 5 parts, &c., is written $20^f \,.\, 8' \,.\, 5''$. &c.

The operation employed to compute superficial and solid contents is that of Multiplication, conducted by means of a mixed *Decimal* and *Duodecimal* scale of Notation; the figures of the feet being expressed and multiplied in the ordinary way, whilst in the other places, the number 12 is always made use of instead of 10. The denomination on the left hand of the multiplier is used first, those of the multiplicand being taken as usual; then the next in order, and so on: and for the reason that we put the first figure of a partial product one place to the *left* of that of the preceding one, when we begin with the *least* denomination of the multiplier, the terms of product here must each be put one place to the *right* of those of the preceding, in order to possess their proper relative values: and the addition is effected by beginning at the lowest denomination, as in compound quantities.

From the circumstance above mentioned, the process is sometimes called *Cross Multiplication;* and it is also frequently termed *Duodecimal* Multiplication or *Duodecimals:* but these names are evidently misapplied, because the *different* digits of the various denominations are not connected with each other by the number 12, though the *denominations* themselves are. The practical applications of the rule will be best taught by examples.

Ex. 1. Find the area of a rectangular parallelogram whose adjacent sides are 5 ft. 3 in., and 4 ft. 9 in.



Here, $AB = 5^f . 3'$, and $AD = 4^f . 9'$:
whence, the area $= 5^f . 3' \times 4^f . 9'$: and the multiplication is effected in the following form:

$$\begin{array}{ll}
5^f \, . \; 3' & = \text{length:} \\
4 \; . \; 9 & = \text{breadth:} \\
\hline
21 \; . \; 0 & = \text{product by } 4^f\text{:} \\
\;\; 3 \; . \; 11 \; . \; 3 & = \text{product by } 9'\text{:} \\
\hline
24^f \, . \; 11' \, . \; 3'' & = \text{area:}
\end{array}$$

and precisely as in a product in the common scale of notation, the denomination of $11'$ is a *twelfth* part of a square foot, which is called a *superficial prime:* that of $3''$ is a twelfth part of a superficial prime, termed a *superficial second:* and so on, if there were more terms: so that the area expressed in square feet is

$$24 + \frac{11}{12} + \frac{3}{144} = 24 + \frac{135}{144} = 24 \text{ sq. feet, } 135 \text{ sq. inches.}$$

This result may easily be verified by either *Vulgar Fractions* or *Decimals:* thus,

in vulgar fractions, the area $= 5\frac{1}{4} \times 4\frac{3}{4} = \frac{21}{4} \times \frac{19}{4} = \frac{399}{16}$

$= 24\frac{15}{16} = 24\frac{135}{144}$ sq. feet, as above:

in decimals, the area $= 5.25 \times 4.75 = 24.9375$

$= 24$ sq. feet, 135 sq. inches, as before.

Ex. 2. Required the area of a square whose side is $7^f \, . \, 8' \, . \, 9''$.

The operation here requisite will be the following:

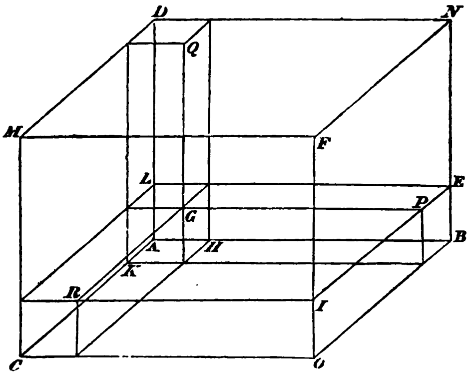
$$\begin{array}{r}
7^f . \; 8' \; . \;\; 9'' \\
7 \; . \; 8 \; . \;\; 9 \\
\hline
54 \; . \; 1 \; . \;\; 3 \qquad\qquad \\
5 \; . \; 1 \; . \; 10 \; . \; 0 \qquad \\
5 \; . \;\; 9 \; . \; 6 \; . \; 9 \\
\hline
59^f . \; 8' \; . \; 10'' \; . \; 6''' \; . \; 9''''
\end{array}$$

or, the area is equivalent to 59 feet, 8 primes, 10 seconds, 6 thirds and 9 fourths, all superficial measure: and expressed in square feet as above, it will evidently be

$$59 + \frac{8}{12} + \frac{10}{144} + \frac{6}{1728} + \frac{9}{20736} = 59\,\frac{15345}{20736} = 59\,\frac{1705}{2304}$$

square feet: and the square inches, square parts, &c., might be found by the ordinary reductions.

Ex. 3. Find the content of a parallelopiped whose lineal dimensions are 5 feet 6 inches, 4 feet 5 inches, and 3 feet 4 inches.



Here, $AB = 5\text{ft. } 6\text{in.}$, $AC = 4\text{ft. } 5\text{in.}$, $AD = 3\text{ft. } 4\text{in.}$:

$$\begin{array}{rrrr}
5^f. & 6' & & \\
4\,. & 5 & & \\
\hline
22\,. & 0 & & \\
2\,. & 3 & .\,6 & \\
\hline
24\,. & 3 & .\,6 & \\
3\,. & 4 & & \\
\hline
72\,. & 10 & .\,6 & \\
8\,. & 1 & .\,2 & .\,0 \\
\hline
80^f. & 11' & .\,8'' & .\,0''': \\
\hline
\end{array}$$

or, the content is 80 solid feet, 11 solid primes, and 8 solid seconds, which expressed in solid feet, is therefore

$$80 + \frac{11}{12} + \frac{8}{144} = 80\,\frac{140}{144} = 80\,\frac{1680}{1728}$$

$= 80$ solid feet, 1680 solid inches.

Ex. 4. Required the capacity of a cube, the length of whose edge is 2 feet 9 inches.

The capacity $= 2^f. 9' \times 2^f. 9' \times 2^f. 9' = 7^f. 6'. 9'' \times 2^f. 9'$ $= 20^f. 9'. 6''. 9''' = 20 + \frac{9}{12} + \frac{6}{144} + \frac{9}{1728} = 20\frac{1377}{1728}$ cubic feet $= 20$ cubic feet, 1377 cubic inches; and it may easily be verified by vulgar fractions or decimals.

200. The method of computation just explained is exceedingly simple, and well adapted to the use of workmen. The reverse operations of division and evolution are not often required, and though they might be conducted on the same plan, it will generally be much easier to express the quantities fractionally or decimally, and then to proceed according to the ordinary methods.

The *Prices* of Artificers' work, being at so much per foot, yard, &c., may of course be calculated by the *Rule of Proportion*, or by *Practice*.

Examples for Practice.

(1) Multiply 14ft. 6in. by 12ft. 7in.
Answer: $182^f. 5'. 6''$.

(2) Multiply 25ft. 7in. by 7ft. 10in.
Answer: $200^f. 4'. 10''$.

(3) Multiply 16ft. 5in. by 12ft. 11in.
Answer: $212^f. 7''$.

(4) Multiply $11^f. 11'$ by $2^f. 3'. 4''$.
Answer: $27^f. 1'. 8''. 8'''$.

(5) Multiply $9^f. 4'. 7''$ by $5^f. 6'. 4''$.
Answer: $51^f. 10'. 4''. 0'''. 4''''$.

(6) Multiply $17^f. 3'. 4''$ by $19^f. 5'. 11''$.
Answer: $336^f. 9'. 6''. 8'''. 8''''$.

(7) Multiply $10'. 3''. 4'''$ by $5'. 0''. 6'''$.
Answer: $4'. 3''. 9'''. 9^{iv}. 8^{v}$.

(8) Multiply $13^f. 2'. 6''$ by $1'. 9''. 10'''$.
Answer: $2^f. 4'''. 7''''$.

(9) Find the square yards, &c., in a plane rectangular surface 15ft. 5in. long, and 9ft. 10in. broad.

Answer: 16yds. $7^f. 7'. 2''$.

(10) How many squares of 100 feet are contained in a floor 19yds. 3in. long, and 9yds. 1ft. 6in. broad?

Answer: 16sq. 31ft. 90in.

(11) Find the cost of a slab, 5ft. 7in. long, and 3ft. 8in. broad, at 3*s*. per square foot.

Answer: £3. 1*s*. 5*d*.

(12) Required the price of the carpet of a room, 18ft. 6in. long, and 14ft. 3in. broad, at 5*s*. per square yard.

Answer: £7. 6*s*. $5\frac{1}{3}$*d*.

(13) What is the value of a piece of building ground, 34ft. 9in. by 26ft. 4in., at 1*s*. per square foot?

Answer: £45. 15*s*. 1*d*.

(14) How much will remain of $43\frac{3}{8}$ square yards of carpet, after covering a room $23\frac{5}{8}$ feet long, and $16\frac{7}{12}$ feet broad?

Answer: 76 square inches.

(15) How many square feet of paper will cover the walls of a room, which is 20ft. 10in. long, 16ft. broad, and 10ft. 8in. high?

Answer: 785 sq. ft. 112 sq. in.

(16) Find the whole surface of a room, 22ft. 5in. long, 18ft. 4in. broad, and 11ft. 8in. high.

Answer: 1772 sq. ft. 112 sq. in.

(17) How many square rods of $272\frac{1}{4}$ feet, are there in a rectangular piece of bricklayer's work whose dimensions are 15ft. and 68ft. 9″?

Answer: $3\frac{3}{4}$ sq. rods.

(18) What is the difference between one area of $3\frac{1}{4}$ feet square, and another of $3\frac{1}{4}$ square feet?

Answer: 7 sq. ft. 45 sq. in.

(19) Find the area of a triangle whose three sides are 2ft. 3in., 3ft., and 3ft. 9in.

Answer: 6 sq. ft. 108 sq. in.

(20) Determine the volume of a cube, whose edge is 3ft. 10in. in length.

Answer: 56 solid ft. 568 solid in.

(21) Find the capacity of a rectangular cistern whose dimensions are 4ft. 6in., 5ft. 7in., and 6ft. 8in.

Answer: $167\frac{1}{2}$ cubic feet.

(22) Find the side of a cube which contains 15 solid feet and 1080 solid inches.

Answer: 2 ft. 6 in.

(23) The area of one side of a cube is $12^f.\ 3'$; find its capacity.

Answer: 42 cubic ft. 1512 cubic in.

(24) What length of carpet 2ft. 3in. wide will cover a room 6yds. 1ft. 6in. long, and 5yds. 9in. wide?

Answer: 45 yds. 1 ft. 6 in.

(25) Divide $1532^f.\ 9'\ 9''$ superficial measure, by $81^f.\ 9'$ lineal measure.

Answer: 18 ft. 9 in.

(26) How much in length that is 1 ft. 2 in. broad and $1\frac{1}{2}$ in. thick, will make a solid foot?

Answer: 6 ft. $10\frac{2}{7}$ in.

THE COMPUTATIONS OF GAGERS.

201. Def. The dimensions made use of by *Gagers* are always taken in *inches*, and parts of an inch expressed *decimally*: and from them, the contents of cisterns, malt-bins, &c., are computed by such rules as have been already laid down, which will consequently be expressed in cubic inches, and their decimal parts.

202. *Liquids* are always estimated by the imperial *Gallon*, which is equal to 277.274 cubic inches: and therefore, when the content of a vessel has been ascertained in cubic inches, the number of gallons it contains, will be found by dividing it by 277.274.

Ex. What number of gallons are contained in a cistern whose length is 40 inches, breadth 24 inches, and depth 16 inches?

Here, the content $= 40 \times 24 \times 16 = 15360$ cubic inches:

$$\text{whence, the number of gallons} = \frac{15360}{277.274}$$

$= 55.3964$ gals. $= 55$ gals. 1 qt. 1 pt., nearly.

203. *Malt, Corn*, &c., are always estimated by the imperial *Bushel*, consisting of 2218.192 cubic inches; and

consequently the number of bushels will be obtained by dividing the content, ascertained as before, by 2218.192.

Ex. If a circular room, 5 feet in radius, be filled with malt to the depth of 6 inches: find the number of bushels it contains.

Here, the content $= 3.14159 \times 60^2 \times 6$

$= 67858.344$ cubic inches:

and the number of bushels $= \dfrac{67858.344}{2218.192}$

$= 30.5917$ bush. $= 30$ bush. $2\frac{1}{3}$ pks., nearly.

Whenever the depth is *one* inch, the *content* of any upright vessel or cistern is expressed by the *area* of its surface: and in this sense the term *surface* is sometimes used in gaging.

204. To compare the old liquid and dry measures lately used, with the imperial measures, we have

the new imperial gallon = 277.274 cubic inches:
the old wine gallon = 231.000 . . . .
the old ale gallon = 282.000 . . . .

and it is evident that the old measures may be accurately converted into the new imperial measures, and *vice versâ*, by the *Rule of Proportion.*

Again,

the new imperial bushel = 2218.192 cubic inches:
the old corn gallon = 268.800 . . . .
the old corn bushel = 2150.400 . . . .

and either of these measures may be changed into the other, by the same means.

The following rules will furnish approximations sufficiently near to the *truth,* for all *practical* purposes.

To reduce old wine gallons to imperial gallons, multiply by $\frac{5}{6}$.

To reduce old ale gallons to imperial gallons, multiply by $\frac{59}{58}$.

To reduce old corn bushels to imperial bushels, multiply by $\frac{32}{33}$.

It follows immediately that the imperial measures may be converted into the old measures, if necessary, by inverting the respective multipliers.

Examples for Practice.

(1) How many gallons are contained in a cubic foot?

Answer: 6.232 gallons, nearly.

(2) The length of a cistern is 169 inches, and the breadth is 125 inches: how many gallons does it contain, the liquor being 4 inches deep?

Answer: 304.752 gallons, nearly.

(3) What is the content of a cylindrical vessel, the radius of whose base is 20 inches, and height 54 inches?

Answer: 244.733 gallons, nearly.

(4) How many bushels of malt are there on a floor $5\frac{1}{2}$ feet by 4 feet, when its depth is 14 inches?

Answer: 20 bushels, nearly.

(5) The diameter of the base of a standard bushel measure is 18.789 inches: find its height.

Answer: 8 inches, nearly.

(6) What number of bushels are contained in the space of a cubic yard?

Answer: 21.033 bushels, nearly.

(7) How many imperial gallons are equivalent to a hogshead of wine, old measure?

Answer: 52.486 gallons, nearly.

(8) Required the number of imperial bushels equivalent to an old or *Winchester* quarter.

Answer: 7.755 bushels, nearly.

The practical calculations of *Excisemen* are greatly facilitated by means of an instrument called a *Sliding Rule*, and by *Tables* containing the proper multipliers and divisors for *squares, circles*, &c., which may be seen in any work treating expressly upon the subject.

THE COMPUTATIONS OF LAND-SURVEYORS.

205. DEF. The dimensions of land, or of any surface of considerable extent, are taken by means of *Gunter's*

Chain, which is 4 *poles* or 22 *yards* in length, and is divided into 100 equal parts called *Links.*

206. Since, an acre is equal to a rectangular parallelogram, 40 poles or 10 chains in length, and 4 poles or 1 chain in breadth, it will contain $1000 \times 100 = 100000$ square links; and therefore, if the lineal dimensions be expressed in links, and the superficial contents be found, these results when divided by 100000, or with *five* figures cut off towards the right hand, will give the numbers of acres, and parts of an acre expressed in decimals.

A lineal pole being $5\frac{1}{2}$ yards or 25 links, the magnitude of a square pole will be

$$5\tfrac{1}{2} \times 5\tfrac{1}{2} = 30\tfrac{1}{4} \text{ sq. yards; or, } 25 \times 25 = 625 \text{ sq. links:}$$

so that we have, in the mensuration of land,

a pole = 625 sq. links, or 30.25 sq. yds.:

a rood = 25000 sq. links, or 1210 sq. yds.:

an acre = 100000 sq. links, or 4840 sq. yds.

Hence also, the magnitude of a square mile

$$= 1760 \times 1760 = 3097600 \text{ sq. yds.} = 640 \text{ acres.}$$

A *Hide* of land sometimes mentioned in old documents, is 100 acres: also, a *Yard* of land is 30 acres.

Ex. The length of a rectangular field being 25 chains 8 links, and its breadth 14 chains 75 links: what number of acres does it contain?

$$\begin{array}{rrr}
\text{Here, 25 chains 8 links} = & 2508 & \text{links,} \\
\text{14 ... 75 ...} = & 1475 & \text{...} \\
\hline
 & 12540 & \\
 & 17556 & \\
 & 10032 & \\
 & 2508 & \\
\hline
\text{acres} & 36.99300 & \\
 & 4 & \\
\hline
\text{roods} & 3.97200 & \\
 & 40 & \\
\hline
\text{poles} & 38.88000 & \\
\hline
\end{array}$$

and therefore the field comprises 36 ac. 3 ro. $38\frac{22}{25}$ po.

Hence also, if the length of a rectangular field containing 36 ac. 3 ro. $38\frac{22}{25}$ po., be 25 chains 8 links, its breadth will be found by reversing the operation upon these magnitudes when expressed in links: thus,

$$\text{the breadth} = \frac{3699300}{2508} = 1475 \text{ links} = 14 \text{ chains } 75 \text{ links.}$$

Examples for Practice.

(1) Find the area of a square field whose side is $10\frac{1}{2}$ chains.

Answer: 11 ac. 4 po.

(2) The base of a triangular field is 16 chains 3 poles, and its perpendicular is 6 chains 2 poles: what number of acres does it contain?

Answer: 5 ac. 1 ro. 31 po.

(3) The sides of a triangular field are 380, 420 and 765 yards: how many acres are contained in it?

Answer: 9 ac. 38 po., nearly.

(4) The diagonal of a trapezium is $5\frac{1}{4}$ chains, and the two perpendiculars upon it from the opposite angles are 3 and $2\frac{1}{2}$ chains: find its area.

Answer: 1 ac. 1 ro. 31 po.

(5) A field in the form of an ellipse has its greatest and least diameters equal to 7 and 5 chains; find how many acres there are in each of the parts into which they divide it.

Answer: 2 ro. 30 po., nearly.

(6) Two acres of land are to be cut from a rectangular field whose breadth is 2 chains 50 links, by a line parallel to it: find the length of the plot.

Answer: 8 chains.

(7) What is the length of the side of a square field comprising 2 acres 4 poles?

Answer: $4\frac{1}{2}$ chains.

(8) The base of a triangular field is 11.313708 chains: find the length of the line parallel to the base, which divides it into two equal parts.

Answer: 8 chains.

IMPERIAL WEIGHTS AND MEASURES.

207. The Weights and Measures made use of in this Kingdom having, from length of time, become subject to certain irregularities, in addition to the want of uniformity which generally prevailed in them, and the *Standard* Weights and Measures being at best but vaguely defined, the subject was at length laid before a Board of Commissioners: and, in accordance with a report furnished by them, *An Act of Parliament*, which came into operation on the first of January 1826, was passed, establishing an uniform System of *Imperial Weights* and *Measures*, the leading features of which we shall now endeavour to place before the reader.

208. Def. The *Standard* of Lineal Measure is a rod or beam whose length is called a *yard*, which is equivalent to 3 *feet* or 36 *inches*: and the Standard *Square* and *Cubic* Measures will therefore depend entirely upon it, as has been seen in the preceding pages.

At present we have no means of ascertaining why this particular length was originally fixed upon; but as it is most essential that it should always remain the same, it will be found convenient to refer it to something else, which we have no reason to suppose ever undergoes any change.

Now the length of a *Pendulum* vibrating *seconds*, or performing 86400 oscillations in the interval between the Sun's leaving the meridian of a place and returning to it again, is always the same at a *fixed* place and under the *same* circumstances: and if this length be divided into 391392 equal parts, the yard is *defined* to be equivalent to 360000 of these parts: also, conversely, since a yard is equal to 36 inches, it follows that the length of the seconds' pendulum expressed in inches, is 39.1392.

The pendulum referred to in this country, is one vibrating seconds at *Greenwich* or in *London*, at the level of the sea in a non-resisting medium: and should the *standard yard* at any time be lost or destroyed, it would be easy to have recourse to experiment for its recovery.

The standard yard being the general *Unit* of lineal measure, it follows that all lengths less than a yard will be expressed by fractions: and it is on this account that a lineal *Inch*, or *ten thousand* of the aforesaid portions

of the pendulum, is conveniently adopted as the *unit* of lineal measure when applied to small magnitudes.

Hence also, by the same means, the standard superficial and solid measures will be accurately ascertained and kept correct.

209. DEF. The *Imperial Gallon* is the standard unit of the measure of capacity, and is *defined* to be 277.274 cubic inches, the lineal inch being that above determined. The gallon, and its multiples and parts, are used to measure both *liquids*, as Water, Spirits, &c., and *dry* goods, as Malt, Corn, &c., and the system is therefore termed *The Imperial Liquid* and *Dry Measure.*

The *Imperial Bushel*, consisting of *eight* gallons, will consequently be 2218.192 cubic inches, and the *form* of the measure is defined by *Act of Parliament:* it is to be an upright cylinder, whose internal diameter is 18.789 inches, and depth 8 inches: but this can be of no importance whatever, when *heaped* measures are once abolished.

The *Act of Parliament* directs that the *Heaped Imperial Bushel* shall be an upright cylinder whose diameter is not less than *twice* its depth, and that the height of the conical heap shall be at least *three fourths* of the depth, the boundary of its base being the outside of the measure: also, in heaped measure, it is ordered that

3 *Bushels* make 1 *Sack*,

12 *Sacks*........ 1 *Chaldron:*

and the bushel is to be equal to 2815.4887 cubic inches.

210. DEF. The *Imperial Pound Avoirdupois*, which is the standard unit by means of which all heavy goods of large masses are weighed, is *defined* to be the weight of one *tenth* part of an imperial gallon, or of 27.7274 cubic inches of distilled water, ascertained at a time when the barometer stands at 30°, and the height of *Fahrenheit's* thermometer is 62°: and this standard may consequently be verified or recovered at any time, when it may be necessary to appeal to experiment.

If the weight of a cubic inch of distilled water be divided into 505 equal parts, and each of such parts be defined to be a *Half Grain*, it follows that 27.7274 cubic inches contain very nearly 7000 such grains; and it is

hence declared by *Act of Parliament* that 7000 *Grains* exactly shall hereafter be considered as the *Pound* Avoirdupois: and that 10 *Grains* shall be equivalent to 1 *Scruple,* and 3 *Scruples* to 1 *Dram:* but these latter denominations are seldom necessary unless great nicety be required.

This weight receives its name from *Avoirs* the ancient name for *goods* or *chattels,* and *Poids* signifying *weight,* in the ordinary language of the country at the time of the *Normans.*

211. DEF. The *Imperial Pound Troy* is *defined* to be 5760 grains, determined as in the last article: and this weight, we are told in the "Report of the Commissioners of Weights and Measures," is retained, because all the Coinage has been uniformly regulated by it, and all Medical Prescriptions or Formulæ now are, and always have been estimated by Troy Weight, under a peculiar subdivision, which the *College of Physicians* have expressed themselves most anxious to preserve.

It is also stated, that there are reasons to believe that the word *Troy* has been derived from the Monkish name given to *London,* of *Troy-novant,* founded on the Legend of *Brute,* which is mentioned by many of the old English writers. The story avers that *Brute,* a lineal descendant of *Æneas,* about the year of the world 2855, founded the city of *London,* then called *Trinovantum,* which was afterwards corrupted into *Trenovant* or *Troynovant.*

212. In conformity with the regulations above mentioned, the Avoirdupois and Troy pounds are easily compared: for it is evident that 5760 lbs. avoirdupois are equivalent to 7000 lbs. troy: and therefore we have

$$1 \text{ lb. avoirdupois} = \frac{7000}{5760} \text{ lbs. troy} = \frac{175}{144} \text{ lbs. troy} = 14 \text{ oz.}$$

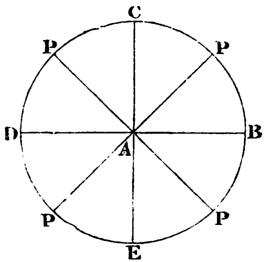
11 dwts. 16 grs. troy, as reduced according to the table:

$$\text{also, } 1 \text{ lb. troy} = \frac{144}{175} \text{ lbs. avoirdupois} = 13 \text{ oz. } 2 \text{ dr. } 1 \text{ scr.}$$

$9\frac{19}{35}$ grs. avoirdupois, by a similar process:

and the subdivisions of each pound may be compared in the same manner.

213. Def. The *Unit* of *Angular Measure* is called a *Degree*, and is written or marked 1°: thus,



if the right angle *BAC* be supposed to be divided into 90 equal parts, each of them is called a degree, and is considered to be the angular unit: and if the angle *BAP* be equal to any assigned number of such parts, as for instance 45, this angle will be 45° on the same scale that the right angle *BAC* is 90°: and thus the magnitudes of any two or more angles may be compared.

If with the angular point *A* as a centre, and any assumed radius *AB*, a circle be described, and the diameters *BAD, CAE* be drawn at right angles to each other, dividing the circumference into four equal parts, *BC, CD, DE, EB*, called *Quadrants;* then it is evident, that the arcs *BC* and *BP* will have the same ratio to one another as the angles *BAC, BAP*, which they respectively subtend; and it is on this principle that a quadrantal arc is said to contain 90 degrees, and therefore the whole circumference to be equal to 360 degrees; and although an angle and an arc, being *heterogeneous* magnitudes, cannot have the same unit of measure, it is clear that the division of the right angle into 90 equal portions of *angular measure* will correspond to the division of the quadrantal arc into 90 equal portions of *lineal measure.*

Thus, then, a degree of the right angle, and a degree of the quadrantal arc are entirely *different* things; but being always *proportional* to each other, they will be connected by an *invariable* factor, when the radius is given, so that either of them being known, the other may be immediately ascertained. That is, if the radius of the circle be the lineal unit or 1,

the whole circumference $= 2 \times 3.14159 = 6.28318$, nearly,

and the quadrantal arc $= 1.57079$, nearly:

whence, we have,

$$\angle BAC : \angle BAP :: \text{arc } BC : \text{arc } BP,$$

$$\text{or, } \angle BAP = \angle BAC \times \frac{\text{arc } BP}{\text{arc } BC}$$

$$= \frac{90^\circ}{1.57079} \times \text{arc } BP, \text{ nearly,}$$

$$= 57^\circ 2957795 \times \text{arc } BP, \text{ nearly:}$$

and therefore the magnitude of the $\angle BAP$ is found by multiplying $57^\circ 2957795$, by the number of *lineal* units contained in the corresponding arc:

$$\text{also, } 57^\circ 2957795 = 57^\circ.17'.44''.48''', \text{ nearly,}$$

must consequently be regarded as the *unit* of angular measure in all cases where an angle is to be determined by means of the number of lineal units in its arc; or, which is the same thing, by the number of radii to which its arc is equivalent.

THE CALENDAR.

214. Def. The interval of time between two passages of the Sun across the meridian of any place, when taken at its *mean magnitude*, is termed a *Day* or a *Mean Solar Day*, which is supposed to be divided into 24 equal portions called *Mean Solar Hours*. And it appears, from the observations and calculations of Astronomers, that the time between the Sun's leaving a fixed point in his path called the *Ecliptic*, and returning to it again, consists of 365.242264 such days, or of 365 days, 5 hours, 48 minutes, $51\frac{3}{5}$ seconds very nearly, which is therefore termed a *Solar Year:* and thus the solar year, as here defined, does not consist of an exact number of solar days, but is always expressed in days by a mixed fractional quantity.

For the purposes of civil life, it would be exceedingly inconvenient that one year should commence at one time of the day, and another at a different time: and this circumstance soon gave rise to the invention of the *Civil Year*, which we shall now endeavour to explain.

215. When the Science of Astronomy was much less perfect than it is at present, the length of the solar year was much less accurately known; and accordingly we find that in the time of *Julius Cæsar*, it was supposed to consist of 365 days, 6 hours, or of $365\frac{1}{4}$ days, *exactly*. On this supposition, it is evident that if, out of *four* years in succession, any *three* consisted of 365 days each, and the remaining one of 366 days, the Sun would have returned, at the end of these *four* years, to the place in the ecliptic which it occupied at their commencement.

The scheme was therefore called the *Julian Calendar;* and if the hypothesis had been correct, it would have been attended with much convenience: the additional day was made by *repeating* the *Sixth* of the *Calends of March* in the Roman Calendar, which corresponds with the 24th of *February* in ours: also, the *year* in which it was inserted was termed *Bissextile*, and the additional *day* was called *Intercalary*, on that account.

This regulation, applied to the years of the *Christian Æra*, was so conducted that, whenever the number of years was divisible by 4, the corresponding year consisted of 366 days, and was called *Leap-year*, the month of *February* having 29 days in that year, and in each of the remaining three years only 28 days, without interfering at all with their order.

Hence also, the remainder after the division of any other number of years by 4, was the number of years since a leap year occurred up to that year: thus, in the year 1839 this remainder is 3; and accordingly it is 3 years since the last leap year happened, and it is 1 year before the next will occur, according to this scheme.

216. Since the true solar year is 365.242264 days, and not 365.25 days, it is evident that the reckoning of time, according to the Julian Calendar, would place the end of the year *after* the time when the Sun had returned to the point of the ecliptic occupied by it at the beginning of the year, and consequently in *advance* of the course of the *seasons:*

but, the error in one year is

$$365.25 - 365.242264 = .007736 \text{ of a day:}$$

whence, by finding how often this is contained in 1 day,

$$\frac{1}{.007736} = 129.2657, \text{ nearly,}$$

will be the number of years in which the error amounts to 1 day: also, by the rule of proportion,

129.2657 yrs. : 400 yrs. :: 1 day. : 3.0944 days, nearly:

whence, it follows that 3.0944 days, or 3 days, 2 hours, 16 minutes, is very nearly the error which would accumulate in 400 years.

Now, according to the Julian Calendar, 400 years would comprise 100 leap years; and since we find that this reckoning falls nearly 3 days *after* the true time, it is evident that if there were only 97 leap years in the same space of time, the year corresponding would very nearly agree with the true solar year; and it is accordingly ordained that whenever the *numbers* expressing the *Centuries*, as 16, 17, 18, 19, &c., denoting 1600, 1700, 1800, 1900, &c., are *not* divisible by 4, the corresponding year shall *not* be a leap year, although according to the Julian Computation it would: thus, 1600 would be a leap year, but 1700, 1800, 1900, would not.

The calendar thus corrected, though not absolutely accurate, is very well adapted to every practical purpose, as the error in 5000 years will not amount to much more than twenty-eight hours. This correction was first promulgated in Europe by *Pope Gregory* in the year 1582, and the calendar has since been called the *Gregorian Calendar*, but it was not introduced into *Protestant Countries* till a much later period. In *England*, it was adopted on the second day of September 1752, when the error amounted to 11 days: and it is called the *New Style*, to distinguish it from the *Julian Calendar*, which is now termed the *Old Style*.

Had the old style continued, the error would now have been 12 days, because 1800 would, according to it, have been a leap year, which in the new it was not: and thus, we have in Almanacks, *Old Christmas-day*, *Old Midsummer-day*, &c., taking place 12 days after the times in which they are fixed by our present system.

Though all the calculations of modern times are conducted by means of the new style, a knowledge of

the difference of the two styles is not without its use, both in the perusal of old Documents, and in the Astronomical Verification of Historical Facts, which could not be performed without it.

217. The common civil year thus fixed and determined, is then subdivided into twelve Calendar Months, as described in the Table. The word *Month* however, is frequently used in different senses: sometimes to denote a *twelfth* part of the year or $30\frac{5}{12}$ days; sometimes as equivalent to 4 weeks or 28 days; and accordingly, a year is said to be equivalent to 13 months and 1 day, or to 52 weeks and 1 day, with the addition of another day when it happens to be leap year.

FRENCH IMPERIAL MEASURES, &c.

218. In consequence of the irregularity in the measures and multiples of all the units just mentioned, it is evident that the calculation of measures and weights will be much more complicated and difficult, particularly to *Foreigners,* than if they were connected by some common divisor and multiplier; and it was with the view of obviating this inconvenience, that a *New System* of measures and weights has been adopted in *France.*

219. In this system, the length of the Terrestrial Arc from the Equator to the Pole in the Meridian of *Paris,* is taken as the *General Standard;* and the following Synopsis of French Measures exhibits them as compared with the standard of this country.

Lineal Measure.

Millimetre . .	=	.03937	English Lineal Inches:
Centimetre . .	=	.39371	. . . .
Decimetre . .	=	3.9371	. . . .
Metre	=	39.371	. . . .
Decametre . .	=	393.71	. . . .
Hecatometre .	=	3937.1	. . . .
Chiliometre. .	=	39371	. . . .
Myriometre .	=	393710	. . . .

Superficial Measure.

Are	=	119.6046	English Superficial Yards:
Decare . . .	=	1196.046	. . . .
Hecatare . .	=	11960.46	. . . .

Solid Measure.

Decistere . .	=	3.5317	English Solid Feet:
Stere	=	35.317	. . . .
Decastere . .	=	353.17	. . . .

Measure of Capacity.

Millilitre . .	=	.06103	English Cubic Inches
Centilitre . .	=	.61028	. . . .
Decilitre . .	=	6.1028	. . . .
Litre	=	61.028	. . . .
Decalitre . .	=	610.28	. . . .
Hecatolitre . .	=	6102.8	. . . .
Chiliolitre . .	=	61028	. . . .
Myriolitre . .	=	610280	. . . .

We may here observe that the *Metre*, or *one ten millionth part* of the Terrestrial Arc, is the Element of lineal measure; the *Are* or Square Decametre, that of superficial measure; the *Stere* or Solid Metre, that of solid measure, and the *Litre* or Cubic Decimetre, that of the measure of capacity.

220. In like manner the Weights belonging to this system, and expressed in English grains, are

Milligramme .	=	.0154	English Grains:
Centigramme .	=	.1544	. . . .
Decigramme .	=	1.5444	. . . .
Gramme . .	=	15.4440	. . . .
Decagramme .	=	154.4402	. . . .
Hecatogramme	=	1544.4023	. . . .
Chiliogramme.	=	15444.0234	. . . .
Myriogramme.	=	154440.2344	. . . .

Here the *Gramme* is the Element, being the weight cubic centimetre of distilled water.

221. The Angular Measures in the same system expressed in English Degrees, are as follow:

Second	. . =	.00009	English Degrees:
Minute	. . =	.009	. . . .
Grade	. . . =	.9	. . . .

Here 100 Grades are consequently equivalent to 90 Degrees in the English scale: and in the inferior denominations, the *Centesimal* scale is uniformly used by the French, where the English proceed according to the *Sexagesimal.*

222. The unit of Value in France is a silver coin called a *Franc*, consisting of $\frac{9}{10}$ths of pure silver and $\frac{1}{10}$th of alloy: and its subdivisions are as follow:

10 Centimes = 1 Decime:
10 Decimes = 1 Franc.

The value of an English pound sterling is equivalent to that of 25.2 francs, very nearly: and thus, the value of 1 franc expressed in English money is

$$\frac{240}{25.2} = 9.5238d. \text{ or } 9\tfrac{1}{2}d., \text{ very nearly.}$$

223. Wherever this system is used, it is evident that the Theory of Decimals, as laid down in the fifth Chapter, will be sufficient for performing all the fundamental operations of Arithmetic, entirely superseding what has been done in the second chapter of this work.

The Student, who may be desirous of prosecuting his enquiries in this very interesting and important subject, is referred to the Articles, *Weights* and *Measures,* in BARLOW'S *Mathematical and Philosophical Dictionary,* and to the last edition of DR. KELLY'S *Universal Cambist*.

PROBLEMS.

224. We will conclude the Application of Arithmetic to Geometry, with the consideration of a few Problems of common occurrence, in the solution of which, the principles explained in this chapter are generally taken for granted.

(1) If two persons, A and B, start at the same time from two towns C and D, distant 300 miles from each other: when and where will they meet, if they travel at the respective rates of 7 and 8 miles an hour?

Since the rate or velocity of A is 7 miles an hour, and the rate or velocity of B is 8 miles an hour, therefore

$$7 + 8 = 15 \text{ miles}$$

is the distance by which they approach each other in 1 hour, or their *relative* velocity: hence we have

$$15\text{mi.} : 300\text{mi.} :: 1\text{hr.} : 20\text{hrs.}$$

or, 20hrs. is the time in which they approach 300 miles towards each other, and therefore meet: therefore the required time is expressed by the whole distance divided by the sum of their velocities or rates per hour.

Also, $7 \times 20 = 140$ miles, the distance travelled by A,

and $8 \times 20 = 160$ miles, the distance travelled by B:

and they meet at 140 and 160 miles from C and D respectively.

Hence also, the distance between them after any assigned interval may be found.

When two motions in a straight line are in opposite directions, the velocity of approach, or the *relative* velocity is equal to the *sum* of the *absolute* velocities.

(2) A, travelling at the rate of 12 miles an hour, starts 15 miles behind B, who travels only 10 miles an hour: find when A will overtake B, and the distance then travelled by each.

Here, the gain of A upon B is $12 - 10 = 2$ miles in 1 hour, which is their relative velocity: whence,

$$2\text{mi.} : 15\text{mi.} :: 1\text{hr.} : 7\tfrac{1}{2}\text{hrs.};$$

and $7\frac{1}{2}$hrs. is the time in which A gains 15 miles upon B, and therefore overtakes him: so that

$$A \text{ has travelled } 12 \times 7\tfrac{1}{2} = 90 \text{ miles,}$$

$$B \text{ has travelled } 10 \times 7\tfrac{1}{2} = 75 \text{ miles:}$$

and the difference of these distances is 15 miles, which is accordant with the enunciation of the problem.

In cases like this, the velocity of approach or *relative* velocity is the *difference* of the *absolute* velocities, and the time is found by dividing the whole distance by it.

The reasoning employed in these two instances is evidently applicable to any *uniform* motions whether in straight lines or curves, provided the distances be measured along the paths described.

(3) Two couriers pass through a place at an interval of 4 hours, travelling at the rates of $11\frac{1}{2}$ and $17\frac{1}{2}$ miles an hour: how far and how long must the first travel, before he is overtaken by the second?

The relative velocity $= 17\frac{1}{2} - 11\frac{1}{2} = 6$ miles:

also, $11\frac{1}{2} \times 4 = 46$ miles $=$ the distance between them, when the second passes through the given place: whence,

as before, we have $\frac{46}{6} = 7\frac{2}{3}$ hours, the time when the second,

after leaving the given place, overtakes the first:

and therefore the first has travelled in $11\frac{2}{3}$ hours, a distance of $11\frac{1}{2} \times 11\frac{2}{3} = 134\frac{1}{6}$ miles: and the second in $7\frac{2}{3}$ hours, a distance of $17\frac{1}{2} \times 7\frac{2}{3} = 134\frac{1}{6}$ miles.

(4) If 252 men in 5 days of 11 hours each, can dig a trench 210 yards long, 3 wide and 2 deep; in how many days 9 hours long, can 24 men dig a trench of 420 yards long, 5 wide and 3 deep?

The solid content of the first trench is $210 \times 3 \times 2 = 1260$ solid feet: and that of the second is $420 \times 5 \times 3 = 6300$ solid feet.

Now, 252 men in 55 hours, dig 1260 solid feet:

whence, 1 man in 55 hours, digs 5 solid feet:

and 24 men in 55 hours, dig 120 solid feet:

therefore, 24 men in $55 \times 52\frac{1}{2}$ hours, dig 6300 solid feet:

and consequently, $\frac{55 \times 52\frac{1}{2}}{9} = 320\frac{5}{6}$ days of 9 hours, is the time required.

(5) If 10 men in 3 days reap a field, the length of which is 1200 feet, and the breadth 800 feet; what is the

length of a field, whose breadth is 1000 feet, which 12 men can reap in four days?

Here, 10 men, in 3 days, reap 1200×800 square feet:

and 1 man, in 3 days, reaps 120×800 square feet:

also, 1 man, in 1 day, reaps 40×800 square feet:

whence, 12 men, in 1 day, reap 480×800 square feet:

and 12 men, in 4 days, reap 1920×800 square feet:

but $1920 \times 800 = 1536000 = 1536 \times 1000$ square feet:

whence, it immediately follows that the required length is 1536 feet.

(6) If a pipe of 6 inches bore, discharge a certain quantity of fluid in 4 hours: in what time will 4 pipes, each of 3 inches bore, discharge twice that quantity?

If 1 denote the quantity of fluid discharged by the first pipe in 4 hours, we have $\frac{1}{4}$ = quantity discharged by it in 1 hour: but the quantities discharged by the pipes are as the areas of their sections, and therefore as the squares of their diameters: whence,

$\frac{1}{4}$: quantity discharged by one of the second set in 1 hour

$$:: 6^2 : 3^2 :: 4 : 1;$$

and therefore the quantity discharged by one of these pipes in 1 hour $= \frac{1}{16}$:

hence, the quantity by 4 such pipes in 1 hour $= \frac{1}{4}$;

and therefore the quantity discharged by these 4 pipes in 8 hours $= 2$, or twice the quantity discharged by the first in 4 hours: that is, 8 hours is the time required.

(7) If a beam which is 10 in. wide, 8 in. deep. and 5 ft. 6 in. long, weigh 8 cwt. 1 qr.: find the length of another beam, the end of which is a square foot, which shall weigh 1 ton.

The volume of the first beam $= 10 \times 8 \times 66$ solid inches, and that of the second beam $= 12 \times 12 \times$ the required length $= 144x$ solid inches, suppose: also, the weights are proportional to the volumes or masses, and therefore we have

$8\frac{1}{4}$ cwt. : 20cwt. :: $10 \times 8 \times 66$ solid in. : $144x$ solid in.;

$$\text{whence, } x = \frac{20 \times 10 \times 8 \times 66}{8\frac{1}{4} \times 144} = 88\tfrac{8}{9}\text{ in.} = 7\text{ ft. } 4\tfrac{8}{9}\text{ in.}$$

(8) If a ball, whose diameter is 2 inches, weigh 5 lbs.: what must be the diameter of another ball of the same substance which shall weigh 78.125 lbs.?

Since the weights are proportional to the volumes, and therefore to the cubes of the diameters, we have

$$5\text{lbs.} : 78.125\text{lbs.} :: 2^3 : (\text{the required diameter})^3:$$

$$\text{whence, (the required diameter)}^3 = \frac{8 \times 78.125}{5} = 125:$$

$$\text{and the required diameter} = \sqrt[3]{125} = 5 \text{ inches.}$$

(9) A rectangular court, the sides of which are 300 feet and 200 feet, has a walk 20 feet wide cut off from it on every side: compute the area of the walk, and of the remaining portion.

The area of the whole court $= 300 \times 200 = 60000$ square feet: also, since the dimensions are diminished 20 feet on *each* side by the walk, the area of the remaining portion

$= (300 - 40) \times (200 - 40) = 260 \times 160 = 41600$ square feet:

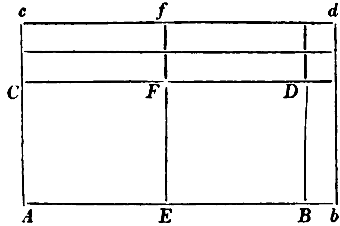
whence, the area of the walk $= 60000 - 41600 = 18400$ square feet: and the walk therefore takes up $\frac{23}{75}$ths of the whole court.

(10) Multiply 2 ft. 1 in. by 1 foot 2 inches: and explain the meaning of the result by a geometrical construction.

Here, by the rule of Article (199), we have

$$\begin{array}{lll} 2^f . & 1' & \\ 1 . & 2 & \\ \hline 2 . & 1 & \\ & 4 . & 2 \\ \hline 2^f . & 5' . & 2'' \\ \hline \end{array}$$

and to explain this result geometrically, if $AB = 2$ feet, $Bb = 1$ inch, $AC = 1$ foot, $Cc = 2$ inches, and the construction be completed as below:



$$\text{we have } Ab = AB + Bb = 2^f.\ 1':$$

$$Ac = AC + Cc = 1^f.\ 2':$$

$$\text{also, } Ad = AD + Db + Dc + Dd:$$

$$\text{but } AD = AB \times AC = 2 \times 1 = 2 \text{ square feet}:$$

$$Db + Dc = 1^f \times 1' + 2^f \times 2' = 1' + 4' = 5 \text{ superficial primes}:$$

$$\text{and } Dd = Bb \times Cc = 1' \times 2' = 2 \text{ square seconds}:$$

that is, the entire product is $2^f.\ 5'.\ 2''$ superficial measure, as before found: and the diagram shews clearly what is meant by each of the denominations of the result; namely, superficial *feet*, *primes* and *seconds*, by means of the parallelograms of *different sizes*.

Hence it appears, that the product of two quantities of the *same* name retains their common *denomination*, whilst the denomination of the product of two quantities of *consecutive* names is the same as that of the *lower*: and this conclusion is sometimes embodied in the form of a *technical* rule.

APPENDIX.

I. NOTATION AND NUMERATION.

It seems probable that the necessities of the human race would at a very early period suggest some method of *counting* or *reckoning*, as well as of *registering* the results of such processes: and the *instruments* employed, which in our language would be called *Counters*, might at any time convey to the mind a very distinct and clear idea of a number which did not consist of *many* individuals. Without entering into any historical account of the different Systems of Notation which have been used in different nations, or hazarding any conjectures as to the circumstances in which they may have had their origin, it is deemed sufficient for our present purpose to pass on immediately to the Notation now in use, which is fully explained in the first chapter of this work.

2. The characters 1, 2, 3, 4, 5, 6, 7, 8, 9, 0, are said to have been transmitted to us from the *Arabians*, who again are supposed to have received them from the *Hindoos*, though in forms considerably different from those in which they are now written: and as before hinted, the word *Digit* usually applied to them, denoting a *Finger*, seems to point out the means originally employed in estimating or computing numerical magnitudes, the number 10, which is called the *Base* or *Radix* of the system, and by which the *local values* of the *digits* are regulated, being that of the *Fingers* of *both* hands. The Notation appears to be as *complete* and *convenient* as can well be imagined, and in its present state may certainly be regarded as one of the greatest and most successful efforts of human ingenuity ever exhibited to the world.

The reader who may be desirous of full information upon this subject is referred to *Professor Leslie's* interesting Work, entitled, *The Philosophy of Arithmetic*.

3. In reference to what was said in Article (14), it may be proper to observe that the method of proceeding differs from that adopted by the *French* and some other *Foreign* Arithmeticians, who adhere throughout to divisions of *three* figures, according to the principle of Article (11), and after the division of *Millions*, proceed directly to that of *Billions*, tens of *Billions* and hundreds of *Billions*: then to *Trillions*, tens of *Trillions*, and hundreds of *Trillions*, and so on: and this method certainly possesses some advantages in point of simplicity; but as numbers of these magnitudes are not of very frequent occurrence, it has not been thought necessary in the present performance to depart from the *Notation* and *Nomenclature* established in this country. In the following schemes it will easily be seen in what respects they differ.

English Nomenclature.

$$\text{\&c.}\quad \overbrace{987654}^{\text{Billions.}}\quad \overbrace{321987}^{\text{Millions.}}\quad \overbrace{654321}^{\text{Units.}}$$

where each division consists of *six* figures: and it may be extended towards the left hand as far as we please.

French Nomenclature.

$$\text{\&c.}\quad \overbrace{987}^{\text{Quadrillions.}}\quad \overbrace{654}^{\text{Trillions.}}\quad \overbrace{321}^{\text{Billions.}}\quad \overbrace{987}^{\text{Millions.}}\quad \overbrace{654}^{\text{Thousands.}}\quad \overbrace{321}^{\text{Units.}}$$

each division consisting of *three* figures: and it is evident that as far as hundreds of millions are concerned, there is no difference whatever in the *reading* or *enumerating* of numbers in the two methods.

II. ADDITION AND SUBTRACTION.

4. The very idea of number implies a capability of *increase* or *decrease*, the former of which is produced by the *operation* of *Addition*, and the latter by that of *Subtraction*: and a set of *Counters* here represented by *units* will be of use in explaining the grounds upon which these operations are established.

Thus, suppose we wish to add *five* and *seven* together, then we have the following *parcels* of counters to represent them:

1, 1, 1, 1, 1, and 1, 1, 1, 1, 1, 1, 1;

and if these be *added* together, or collected into *one* parcel, their sum will evidently be represented by

1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1,

which may again be put in the form,

1, 1, 1, 1, 1, 1, 1, 1, 1, 1,
1, 1;

and this indicates the result of the operation performed to be *ten* together with *two*, or *twelve:* that is, the *sum* of *five* and *seven* is *twelve.*

Hence, in a system of counters there is in fact *no operation* to perform, except merely to *collect* or *combine* the counters into one group: and there would be no necessity for committing to memory the sum of two numbers as in our system, except so far as the *name* of that sum is concerned.

The same might manifestly be done with *more* and *larger* numbers, and it furnishes the *definition* of the operation of *Addition* given in Article (21).

5. To subtract *six* from *nine*, implying that of *nine* individuals, *six* are to be taken away, we must evidently have at first *nine* counters, as

1, 1, 1, 1, 1, 1, 1, 1, 1,

which may be formed into the two *parcels*,

1, 1, 1, 1, 1, 1, and 1, 1, 1;

then if we withdraw *six* of these, or remove the *first* parcel, we have only three counters left, denoted by

1, 1, 1:

and thus we see that if *six* be *subtracted* from *nine*, the *remainder* will be *three.* Here the *withdrawal* of the *less* number of counters, or the *removal* of the *former* parcel, is the only process employed, and of course it

needs no effort of the *mind* to perform it: and our system has always a *tacit* reference to this circumstance, the operation of *Subtraction* being entirely founded upon it, and taking its *definition* from it, as in Article (26).

6. The student will perceive that the *performance* of these two operations is not *facilitated* by the modern notation, except as to the *writing* and *reading* of the results. On the contrary, they are rendered considerably more *difficult*, and require *Rules and Directions* to work by, which have already been laid down in Articles (22) and (27): they however *depend* upon a system of counters, owe their *origin* entirely to it, and may at any time be performed by means of it.

Thus, retaining the use of the arithmetical signs for the operations of Addition and Subtraction, we have

$$3 = 1 + 1 + 1:$$

$$5 = 1 + 1 + 1 + 1 + 1:$$

whence,

$$\begin{aligned} 3 + 5 &= (1 + 1 + 1) + (1 + 1 + 1 + 1 + 1) \\ &= 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 \\ &= 8, \end{aligned}$$

by omitting the brackets, which were introduced merely to keep the two parcels distinct from each other, and representing the *aggregate* or *assemblage* of units by its proper modern symbol: and it is here shewn, if need be, that it is quite immaterial in what *order* the numbers to be added are taken.

Again, $7 = 1 + 1 + 1 + 1 + 1 + 1 + 1:$

$$4 = 1 + 1 + 1 + 1:$$

whence, if from the former of these be withdrawn or removed what is *equivalent* to the latter, there will remain $1 + 1 + 1$ or 3: and we shall have $7 - 4 = 3$.

7. Although in the operations of Addition and Subtraction as treated of in the text, it has been found convenient to commence at the *right* hand and proceed towards the *left*, the use of the arithmetical signs will enable us to perform the same operations in any order

we may choose: thus, to find the sum and difference of 1345 and 274, we have

$$
\begin{aligned}
1345 &= 1000 + 300 + 40 + 5 \\
274 &= \phantom{1000 + {}} 200 + 70 + 4 \\
\hline
\text{and the sum} &= 1000 + 500 + 110 + 9 \\
&= 1000 + 500 + 100 + 10 + 9 \\
&= 1000 + (500 + 100) + 10 + 9 \\
&= 1000 + 600 + 10 + 9 \\
&= 1619:
\end{aligned}
$$

$$
\begin{aligned}
\text{also, } 1345 &= 1000 + 300 + 40 + 5 \\
&= 1000 + 200 + 100 + 40 + 5 \\
&= 1000 + 200 + 140 + 5 \\
274 &= \phantom{1000 + {}} 200 + 70 + 4 \\
\hline
\text{and the difference} &= 1000 + 0 + 70 + 1 \\
&= 1071.
\end{aligned}
$$

From these processes, which have a close resemblance to the method of reckoning by counters, we cannot but see, that by a *slight* exercise of the *mind* and the *memory*, much *real* labour is saved by means of the rules in the text, not to mention the *prolixity* of operation as well as the *number* of figures that would be required for larger magnitudes than those which have been used to establish this conclusion.

III. MULTIPLICATION AND DIVISION.

8. The operation intended by the word *Multiplication*, must necessarily be that which is defined and exemplified in Article (31) of the text, and it will here be required only to shew that the conclusions which it leads to, may be safely depended upon, as far as the *order* of the *factors* may influence the *product*.

To multiply 4 by 3, we have to repeat 4 or $1 + 1 + 1 + 1$, *three times*, and the product will therefore be

$$
\begin{aligned}
&(1 + 1 + 1 + 1) + (1 + 1 + 1 + 1) + (1 + 1 + 1 + 1) \\
&= 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 \\
&= (1 + 1 + 1) + (1 + 1 + 1) + (1 + 1 + 1) + (1 + 1 + 1),
\end{aligned}
$$

which is manifestly $1 + 1 + 1$ or 3, *four times* repeated: that is, *three* times *four* is the same as *four* times *three*.

By reasoning of this kind it is made to appear that the product has a *similar* or *symmetrical* relation to both its factors, because it remains the same when the multiplicand is made the multiplier, and the multiplier becomes the multiplicand.

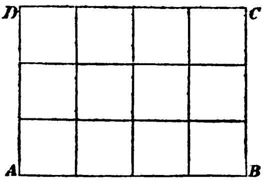
9. If *each* of the *units* be supposed to represent a *man*, or other *specified* individual, the process employed above is evidently no longer admissible, as the result becomes at once unintelligible, and admits of no *numerical* interpretation. That is, the multiplication of *concrete* magnitudes *as such* is altogether impossible: but whenever it is applied to them, they are first supposed to be *divested* of their *concrete* character, or to be *abstracted:* and when the operation has been performed upon the corresponding *abstract* magnitudes, we have only to *explain* or *interpret* when possible, the meaning of the result agreeably to the circumstances of the case.

Thus, if the factors were £7. and £8., we could easily multiply together the abstract numbers 7 and 8, whose product is 56; but the *denomination* of this *result* cannot possibly be ascertained, and the operation is altogether *useless,* if not *absurd.*

Again, to multiply £7. by the abstract number 8, the *correct* method of proceeding will be to consider the 7 abstracted, to find the product of the numbers 7 and 8 or 8 and 7, and the product 56 must then be interpreted in consistency with the nature of the question proposed: and it is seen immediately, that £7. being *repeated* 8 *times,* amounts to £56.: so that whilst the multiplicand may be a *concrete* magnitude, the multiplier must be an *abstract* one, because the *number* of parcels of £7. each, is in no way dependent upon the *species* of the individuals contained in any one of them, but would remain the same were any other species or number of individuals to be repeated *eight* times.

The same observations will of course be applicable when the *species* of the concrete magnitudes are *different;* thus, if a person walk for 5 hours at the rate of 4 miles an hour, we find the product of the numbers 4 and 5 to be 20, which must, from the circumstances of the case, be 20 *miles* the whole distance travelled, and not 20 *hours,* because such an interpretation would necessarily affect the very nature of the things proposed for consideration.

10. Whenever therefore in the *Application of Arithmetic to Geometry*, the product of *two* or *three* numbers, consisting of *feet*, *inches*, &c., has been spoken of, it is always to be carefully borne in mind that a *new* and *totally different* unit has been introduced for its interpretation, and that the *number* of *such* units is merely expressed by the *same figures* as the product formed in the ordinary way: thus,



if $AB = 4$ inches, and $AD = 3$ inches, each of the portions into which they are here divided will be 1 inch: and there are evidently as many small squares or *new* units in the parallelogram as there are *abstract* units in the product of 4 multiplied by 3, or of 3 multiplied by 4, or as there are *old* units in the product of 4 inches multiplied by 3, or of 3 inches multiplied by 4: and this is all that is meant in the computations and comparisons of *geometrical* magnitudes, whenever such modes of expression happen to be used.

The same kind of Diagram is sometimes adopted to explain *geometrically* the operation of Multiplication, which it perhaps does well enough, if it be distinctly recollected that the units of the product have no *assignable* relation whatever to those in either of the factors, except so far as their *numbers* are concerned: and on the same hypothesis it bears out what has been already proved, as to the product remaining the same whichever of the factors may be used as the multiplicand or multiplier.

Division being the reverse of Multiplication, there will be no difficulty in applying what has just been said, to its elucidation.

11. With the use of the proper arithmetical signs, an explanation somewhat similar may be given by means of our common symbols.

Thus, since $3 = 1 + 1 + 1$, we have

$$3 \times 5 = (1 + 1 + 1) \times 5 = 1 \times 5 + 1 \times 5 + 1 \times 5 = 5 + 5 + 5 = 15:$$

also, since $5 = 1 + 1 + 1 + 1 + 1 = 3 + 2$, we have

$$\begin{aligned} 5 \times 3 &= (3 + 2) + (3 + 2) + (3 + 2) \\ &= 3 + 3 + 3 + 2 + 2 + 2 \\ &= 3 + 3 + 3 + 2 + (1 + 1) + 2 \\ &= 3 + 3 + 3 + (2 + 1) + (1 + 2) \\ &= 3 + 3 + 3 + 3 + 3 = 3 \times 5. \end{aligned}$$

Again, because $25 = 7 + 7 + 7 + 4$, we have

$$\begin{aligned} 25 \div 7 &= (7 \div 7) + (7 \div 7) + (7 \div 7) + \text{remainder } 4 \\ &= 1 + 1 + 1 + \text{remainder } 4 \\ &= 3 \text{ the quotient, with the remainder } 4. \end{aligned}$$

12. The rule given in Article (33) of the text directs us to begin the operation of Multiplication with the figure on the *right* hand of the multiplier, but the proper local values of the digits in the partial products may easily be retained if we commence with that on the *left*: thus,

$$\begin{array}{rl} 325 & \\ 457 & \\ \hline 1300 & = \text{product by } 4 \text{ as } 400: \\ 1625 & = \text{product by } 5 \text{ as } 50: \\ 2275 & = \text{product by } 7 \text{ as } 7: \\ \hline 148525 & \end{array}$$

in which we see at once that every digit retains the same local value as it has in the product obtained by the ordinary process, when the requisite ciphers are supplied.

This is the foundation of the remark made in (199) of the text; and by the same method, the Multiplication of Decimals may be effected so as to retain the decimal points of the partial products in the same vertical line.

13. Any number whatever when divided by 9, leaves the same remainder as the sum of its digits, when divided by 9, leaves.

Thus, taking the number 8432, we have

$$\frac{8432}{9} = \frac{8000}{9} + \frac{400}{9} + \frac{30}{9} + \frac{2}{9}$$

$$= 8 \times \frac{1000}{9} + 4 \times \frac{100}{9} + 3 \times \frac{10}{9} + 1 \times \frac{2}{9}$$

$$= 8\left\{111 + \frac{1}{9}\right\} + 4\left\{11 + \frac{1}{9}\right\} + 3\left\{1 + \frac{1}{9}\right\} + \frac{2}{9}$$

$$= 888 + \frac{8}{9} + 44 + \frac{4}{9} + 3 + \frac{3}{9} + \frac{2}{9}$$

$$= 888 + 44 + 3 + \frac{8 + 4 + 3 + 2}{9}:$$

so that the remainder arising from the division of 8432 by 9, is manifestly the same as that which arises from dividing $8 + 4 + 3 + 2$ or 17, which is the sum of its digits, by 9.

Hence, also a number will be divisible by 9, whenever the sum of its digits is divisible by 9: and moreover, if the sum of its digits be subtracted from any number, the remainder will be divisible by 9. The same property holds good of the number 3: and in precisely the same way it may be proved that a number is divisible or not by 2, according as the digit in its *units'* place is divisible or not by 2: that every number when divided by 5, leaves the same remainder as the digit in its units' place, when divided by 5, leaves, and that when a number is divided by 11, the remainder will be the same as that which arises from dividing the difference of the sums of the digits in the odd and even places by 11. See Article (55) of the text.

14. The property of the number 9 just established is the source of a very convenient test for the correctness of the operation of *Multiplication,* known by the name of *casting out the nines.*

Let the multiplicand and multiplier be 742 and 64 respectively: then we have

$$742 = 82 \times 9 + 4, \text{ and } 64 = 7 \times 9 + 1:$$

and since the product of two numbers is, by Article (33) of the text, the same as the sum of the products of either of them, and all the parts of the other, we shall have it here expressed in the following form:

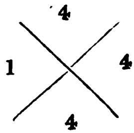
$$742 \times 64 = 82 \times 9 \times 7 \times 9 + 4 \times 7 \times 9 + 82 \times 9 + 4:$$

and this being divided by 9, leaves 4 for a remainder, which is the same as the remainder arising from dividing

by 9, the product of the remainders 4 and 1, which are left after the division of the multiplicand and multiplier respectively, by 9.

The test is usually applied in the following form:

$$\begin{array}{r} 742 \\ 64 \\ \hline 2968 \\ 4452 \\ \hline 47488 \\ \hline \end{array}$$



the 4 on the right hand of the cross being obtained from $7 + 4 = 9 + 2$, $2 + 2 = 0 + 4$: the 1 on the left hand from $6 + 4 = 9 + 1$: the 4 at the top from $4 \times 1 = 0 + 4$: and the 4 at the bottom arises from the product, thus, $4 + 7 = 9 + 2$, $2 + 4 + 8 = 9 + 5$, $5 + 8 = 9 + 4$, the *excess* above 9 in each case being alone considered.

15. It will very easily be seen how the results of the other fundamental operations may be tested by means of the same property: and although the methods recommended in articles (23), (28), (34) and (41) are exceedingly obvious, and devoid of difficulty, it may not be amiss to mention another for *Division* which is of great practical use:

$$\begin{array}{l} 243)723196(2976 \\ 486 \\ \overline{2371} \\ 2187 \\ \overline{1849} \\ 1701 \\ \overline{1486} \\ 1458 \\ \overline{28} \end{array}$$

now, since the *dividend* ought to be equal to the *remainder* together with all the numbers which have been *subtracted* from it, it follows that if we *add* together the *remainder* and all the *partial* products of the divisor, their *sum* should be the same as the dividend: this done, we have here

$$7\ 2\ 3\ 1\ 9\ 6.$$

These tests are all extremely easy of *application*, but not one of them can be depended upon for *absolute* accuracy: thus, in casting out the *nines*, the digits 0 and 9 may be replaced by each other, and the local values of *any* or *all* of the digits may be disarranged whilst the result of the rule remains the same: and in the test last given, the error in one stage of the division may be compensated by another of the *same* magnitude, but *contrary* quality in a subsequent one, a defect to which the one just mentioned is likewise subject.

16. In the operation of *Division*, the number of figures *put down* may be greatly diminished by what is called the *Italian Method*, which omits the partial *subtrahends* and retains only the partial *remainders*: thus,

```
257)39208653(152562
    1350
      658
      1446
        1615
          733
          219
```

and this comprises much fewer figures than the ordinary operation, but it does not furnish the test which has been mentioned in the last article.

Many other contrivances will naturally suggest themselves to the *inventive* student, but what has already been said will generally be sufficient for ensuring some very considerable degree of *practical* correctness.

IV. INVOLUTION AND EVOLUTION.

17. The *first* of these operations, being merely that of Multiplication, is mentioned here, only because the character and circumstances of the *direct* Arithmetical process constitute a *necessary* and *essential* part of the grounds upon which we must endeavour to perform the *inverse* operation of Evolution.

Since the *square* of $28 = 28 \times 28 = 784$, the *square root* of 784 must be 28: and we have to arrive at the latter of these numbers by means of the former: but as

there appears to be no *immediate* connection between them, we shall put 28 in the *form* $20 + 8$, and then determine the corresponding *form* of its square, from the consideration that the product of any two quantities is the sum of the products which arise from multiplying every part of one of them by every part of the other: thus,

$$\begin{array}{rl}
\text{the root is} & 20 + 8 \\
 & \underline{20 + 8} \\
 & 400 + 20 \times 8 \\
 & \qquad\; + 20 \times 8 + 64 \\
\hline
\text{the square is} & \underline{400 + 2 \times 20 \times 8 + 64}
\end{array}$$

which consists of $400 = 20^2$, together with *twice* the product of 20 and 8, and $64 = 8^2$: in order therefore to ascertain the square root of 784, expressed in the form $400 + 2 \times 20 \times 8 + 64$, we first find the square root of 400 to be 20: and then from dividing $2 \times 20 \times 8$ by the *double* of this, or by 2×20, the remaining part 8 of the root is obtained; so that $2 \times 20 + 8$ being *now* made the divisor, and multiplied by 8, and the product subtracted from $2 \times 20 \times 8 + 64$, it appears that the entire root $20 + 8$ or 28 is determined.

Keeping in view the demonstration above given, we may have either of the following operations:

$$\begin{array}{r|l}
\begin{array}{rl}
 & 784\,(20+8 \\
 & \underline{400} \\
2 \times 20 + 8 = 48) & 384 \\
 & \underline{384}
\end{array}
&
\begin{array}{rl}
 & 784\,(28 \\
 & \underline{4} \\
48) & 384 \\
 & \underline{384}
\end{array}
\end{array}$$

the latter being nearly the same as the former, by omitting the *ciphers*, as was done in Multiplication and Division: and we observe that the Rule laid down in Article (159) of the text, is here *investigated* for the particular number under consideration.

Since, the square of 49

$$= 49^2 = (48 + 1)^2 = 48^2 + 2 \times 48 + 1,$$

it is obvious that when the *root* is increased by 1, the corresponding *square* is increased by twice that root + 1: [illegible] same mode of reasoning being equally applicable

in every other instance, it follows that the remainder at any stage of the process can in no case *exceed* the double of the root already obtained: agreeably to the observation made at the end of Ex. 4. of Article (159).

The method of Multiplication used in this and some of the preceding Articles of the Appendix will furnish the means of deriving the square of one number from that of another by a very simple proceeding: thus,

the square of $31 = (30 + 1)^2 = 900 + 2 \times 30 + 1 = 961$:
the square of $53 = (50 + 3)^2 = 2500 + 2 \times 50 \times 3 + 9 = 2809$:
and so on.

18. The rule for the extraction of the cube root given in Article (191), may be investigated in a similar manner, and the observation at the end of the article may be established upon the same principles; but for the reason stated in the text, it will not be necessary to follow up the inverse processes further in this place, inasmuch as they are rendered much clearer by the use of general *Algebraical* symbols, and the rules already laid down are quite sufficient for the performance of the operations in every case that can occur.

V. RATIO AND PROPORTION.

19. The *relation* of two magnitudes may be known by considering how *much* the one is greater or less than the other, or what is their *Difference*, as well as by observing how *many times* the one is contained in the other, or what is their *Quotient*. The former of these views, called *Arithmetical Ratio*, constitutes the chief business of the operation of Subtraction; and the latter is termed *Geometrical Ratio*, because it is generally applied to *Geometrical Magnitudes*, though it derives its importance from the various uses that are made of it in the calculations of civilized life. In which ever way the comparison may be made, it is evident that no *relation* can be established between them unless the magnitudes are of the *same kind;* and consequently *Ratio* as used in the text must be an *abstract* quantity, expressing merely the *numerical* value of one of the magnitudes, with reference to the other considered as *an unit* of the same kind. See Articles (65) and (96) of the text.

From this it follows that the *relation* of any two *concrete* magnitudes of the same kind, as two sums of *Money*, may be the *same as*, or *equal to*, that of two other *concrete* magnitudes of the same kind, as two bales of *Goods:* and this *Equality* of *Ratios* has been defined to be a *Proportion*.

20. It is clearly impossible to institute any such comparison between *Geometrical Magnitudes* without the assistance of their *Arithmetical Representatives*, which it may not always be in our power *accurately* to obtain; and accordingly it is stated in the *fifth* Book of *Euclid's* Elements, that "*Proportion* is the *Similitude of Ratios;* and the *first* of four magnitudes is said to have the same ratio to the *second*, which the *third* has to the *fourth*, when *any equimultiples whatever* of the *first* and *third* being taken, and *any equimultiples whatever* of the *second* and *fourth;* if the multiple of the *first* be greater than that of the *second*, the multiple of the *third* is also greater than that of the *fourth;* if equal, equal; and if less, less."

This conclusion has been established in the text with respect to *numbers* forming a proportion, and it may be applied immediately to shew whether four numbers taken in order constitute a proportion or not. Thus, if

$$2 : 3 :: 4 : 5;$$

by taking equimultiples of the first and third, we have

$$12 : 3 :: 24 : 5;$$

and by taking equimultiples of the second and fourth, we obtain

$$12 : 12 :: 24 : 20,$$

in which the condition above enunciated not being fulfilled, we are assured that the numbers 2, 3, 4, 5 do not form a proportion according to the *geometrical* definition, as the *arithmetical* definition shews at once, because $\frac{2}{3}$ is not *equal to* $\frac{4}{5}$.

THE END.